Mathematical Sciences (MV), Chalmers/GU
Monday, August 15, 2016, 8:30-12:30.

Phone: 5325 Edvin Wedin
Presence of teacher: $\sim 9:30$ and $\sim 11:30$

Tools: Only the attached sheets (2 pages) of formulas. No calculator or handbook is allowed.
(Language Dictionaries are allowed/Språklektion är tillåtet)

Exam in MMG710/TMA362 Fourier Analysis

1. Let $f(\theta)$ be the 2π-periodic function $f(\theta) = |\sin\theta| - \frac{2}{\pi}$. The Fourier series of f is given by

$$f(\theta) = -\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1}\cos n\theta.$$

(**a**) Find the Fourier series of the functions $F(\theta) = \int_0^\theta f(\phi)d\phi$ and $f'(\theta)$.

(**b**) Evaluate the series $\sum_{n=1}^{\infty} \frac{n^2}{(4n^2-1)^2}$. (Hint: Use (a) and Parseval Theorem.)

2. Let $f(x) = \cos^4 x$ and let $V = \text{Span}\{a, b\}$ be the two-dimensional subspace of $L^2(-\pi, \pi)$ spanned by the functions $a(x) = 1$ and $b(x) = \sin^2 x$. Find (**a**) the orthogonal projection of f onto V and (**b**) the distance from f to V.

3. Let $f(x) = \frac{\cos x}{1+x^2}$. Find the Fourier transform $\widehat{f}(\xi)$ and compute the integral $\int_0^\infty f(x)dx$.

4. Solve the following inhomogeneous heat equation

$$\begin{cases} u_t = ku_{xx} + e^t, & t > 0, \quad 0 < x < \pi \\ u_x(0,t) = 0,\ u_x(\pi, t) = 0, & t > 0 \\ u(x, 0) = \cos^3 x, & 0 < x < \pi. \end{cases}$$

5. Find the inverse Laplace transform $f(t) = \mathcal{L}^{-1}[F(z)](t)$ of $F(z) = \frac{1}{z^2-5z-6}$ and find the function $f * f(t)$.

6. Formulate and prove the Bessel's inequality for an orthogonal set $\{\phi_n\}_{n=1}^\infty$ in a general Hilbert space $L^2(a, b)$.

Grades: 6 problems each of 4 points.
MMG710: G (12-17 p.), VG (18-24 p.). **TMA362**: 3 (12-14 p.), 4 (15-17 p.), 5 (18-24 p.)

Basic Fourier Transforms and Laplace Transforms

0.1 Fourier transform $\mathcal{F} : f(x) \mapsto \widehat{f}(\xi)$. ($a > 0$ and $c \in \mathbb{R}$ are constants)

$f(x-c)$	$e^{-ic\xi}\widehat{f}(\xi)$
$e^{i\alpha x}f(x)$	$\widehat{f}(\xi-\alpha)$
$f(ax)$	$a^{-1}\widehat{f}(a^{-1}\xi)$
$f'(x)$	$i\xi\widehat{f}(\xi)$
$xf(x)$	$i(\widehat{f})'(\xi)$
$(f*g)(x)$	$\widehat{f}(\xi)\widehat{g}(\xi)$
$f(x)g(x)$	$(2\pi)^{-1}(\widehat{f}*\widehat{g})(\xi)$
$e^{-a\frac{x^2}{2}}$	$\sqrt{\frac{2\pi}{a}}\, e^{-\frac{\xi^2}{2a}}$
$(x^2+a^2)^{-1}$	$\frac{\pi}{a}e^{-a\lvert\xi\rvert}$
$e^{-a\lvert x\rvert}$	$2a(\xi^2+a^2)^{-1}$
$\chi_a(x)$	$2\xi^{-1}\sin a\xi$
$x^{-1}\sin ax$	$\pi\chi_a(\xi)$

0.2 Laplace transforms $\mathcal{L} : f(t) \mapsto F(z) = \mathcal{L}f(z)$. ($a > 0$ and $c \in \mathbb{C}$ are constants)

$H(t-a)f(t-a)$	$e^{-az}F(z)$
$e^{ct}f(t)$	$F(z-c)$
$f(at)$	$a^{-1}F(a^{-1}z)$
$f'(t)$	$zF(z)-f(0)$
$f''(t)$	$z^2F(z)-zf(0)-f'(0)$
$f*g$	FG
$H*f(t)=\int_0^t f(s)ds$	$z^{-1}F(z)$
$tf(t)$	$-F'(z)$
$t^n e^{ct}$	$\frac{n!}{(z-c)^{n+1}}$
$\sin ct$ resp. $\cos ct$	$\frac{c}{z^2+c^2}$ resp. $\frac{z}{z^2+c^2}$
$\sinh ct$ resp. $\cosh ct$	$\frac{c}{z^2-c^2}$ resp. $\frac{z}{z^2-c^2}$

Some formulas in Fourier analysis

Trigonometric identities

$$e^{ix} = \cos x + i\sin x, \qquad \cos x = \frac{e^{ix}+e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix}-e^{-ix}}{2i},$$
$$\cos(x+y) = \cos x\cos y - \sin x\sin y, \qquad \sin(x+y) = \sin x\cos y + \cos x\sin y,$$
$$\sin^2 x = \frac{1-\cos 2x}{2}, \qquad \cos^2 x = \frac{1+\cos 2x}{2},$$
$$\sin x\ \sin y = \frac{\cos(x-y)-\cos(x+y)}{2}, \qquad \cos x\ \cos y = \frac{\cos(x-y)+\cos(x+y)}{2},$$
$$\sin x\ \cos y = \frac{\sin(x+y)+\sin(x-y)}{2}.$$

Hyperbolic functions

$$\cosh x = \frac{e^x+e^{-x}}{2}, \qquad \sinh x = \frac{e^x-e^{-x}}{2}.$$

Characteristic and Heaviside functions

$$\chi_a(x) = \begin{cases} 1, & |x| < a, \\ 0, & |x| > a. \end{cases}$$

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & \text{else}. \end{cases}$$

Gamma function $\Gamma(a)$

$$\Gamma(a) = \int_0^\infty e^{-x}x^{a-1}dx, \qquad a > 0$$
$$\Gamma(a+1) = a\Gamma(a), \qquad \Gamma(n+1) = n!$$