MATEMATIK
Göteborgs universitet

Some useful formulas for curves

Consider a unit speed curve $\boldsymbol{\gamma}(t)$. Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the Frenet trihedron, with $\mathbf{t} = \gamma'(t)$ the tangent vector, $\mathbf{n}$ the normal vector and $\mathbf{b}$ the binormal.
The *Frenet-Serret equations* are

$$\begin{aligned} \mathbf{t}' &= && \kappa\mathbf{n} && \\ \mathbf{n}' &= -\kappa\mathbf{t} && && + \tau\mathbf{b} \\ \mathbf{b}' &= && -\tau\mathbf{n} && \end{aligned}$$

Here the curvature $\kappa = ||\boldsymbol{\gamma}'||$; if $\kappa \neq 0$, the binormal $\mathbf{b}$ and the torsion τ are defined.

For a general parametrisation the following formulas hold:

$$\kappa = \frac{||\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''||}{||\boldsymbol{\gamma}'||^3}$$

$$\tau = \frac{\det(\boldsymbol{\gamma}', \boldsymbol{\gamma}'', \boldsymbol{\gamma}''')}{||\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''||^2}$$

For a plane curve $\tau \equiv 0$. In that case we can define a signed unit normal vector $\mathbf{n}_s$, such that $\{\mathbf{t}, \mathbf{n}_s\}$ is a positively oriented orthonormal base. Then the equation $\boldsymbol{\gamma}'' = \kappa_s \mathbf{n}_s$ defines the signed curvature, if $\boldsymbol{\gamma}$ is unit speed. The Frenet-Serret equations become

$$\begin{aligned} \mathbf{t}' &= && \kappa_s\mathbf{n}_s \\ \mathbf{n}_s' &= -\kappa_s\mathbf{t} && \end{aligned}$$

For an arbitrary parametrisation $\boldsymbol{\gamma} : (\alpha, \beta) \to \mathbb{R}^2$ the signed curvature is given by

$$\kappa = \frac{\det(\boldsymbol{\gamma}', \boldsymbol{\gamma}'')}{||\boldsymbol{\gamma}'||^3}$$