Lösningar till tentamensskrivningen

MMG720, Differentialgeometri, 20170531

1. **Viviani's curve is the intersection of a sphere of radius 2 and a circular cylinder of radius 1 passing through the origin. It can be parametrised as**

$$\boldsymbol{\gamma}(t) = (1 + \cos 2t, \sin 2t, 2\sin t)$$

with $-\pi \le t < \pi$. Compute the curvature and torsion of this curve.

We compute

$$\begin{aligned} \boldsymbol{\gamma}'(t) &= (-2\sin 2t, 2\cos 2t, 2\cos t)\ , \\ \boldsymbol{\gamma}''(t) &= (-4\cos 2t, -4\sin 2t, -2\sin t)\ , \\ \boldsymbol{\gamma}'''(t) &= (8\sin 2t, -8\cos 2t, -2\cos t)\ . \end{aligned}$$

Then $\|\boldsymbol{\gamma}'\|^2 = 4(1+\cos^2 t)$,
$\boldsymbol{\gamma}' \times \boldsymbol{\gamma}'' = (-4\cos 2t \sin t + 8\sin 2t \cos t, -8\cos 2t\cos t - 4\sin 2t \sin t, 8)$,
$\|\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''\|^2 = 16\sin^2 t + 64\cos^2 t + 64 = 80 + 48\cos^2 t$,
$\det(\boldsymbol{\gamma}', \boldsymbol{\gamma}'', \boldsymbol{\gamma}''') = 48\cos t$.
So

$$\begin{aligned} \kappa &= \frac{\sqrt{5 + 3\cos^2 t}}{2(1+\cos^2 t)^{\frac{3}{2}}} \\ \tau &= \frac{3\cos t}{(5 + 3\cos^2 t)} \end{aligned}$$

2. **Prove the Four Vertex Theorem: every convex simple closed curve in $\mathbb{R}^2$ has at least four vertices.**

If the function κ_s is not constant, it attains its maximum and minimum, say in P och Q. Assume P and Q are the only vertices. The segment PQ, which we may assume to lie on the x-axis, divides the curve in two parts. On one of it $\kappa_s' > 0$, on the other $\kappa_s' < 0$; we may assume that $y < 0$ there. Then $\int_C y\kappa_s' ds > 0$. On the other hand, by partial integration

$$\int_C y\kappa_s' ds = -\int_C y'\kappa_s ds = \int_C x'' ds = 0\ ,$$

because $t' = \kappa_s n$, where $t = \binom{x'}{y'}$ and $n = \binom{-y'}{x'}$. This contradiction shows that κ_s' has one more sign change. If there are three sign changes, then there is a fourth.

3. **For a surface patch $\boldsymbol{\sigma} : \mathbb{R}^2 \to \mathbb{R}^4$ of a surface in 4-space one can define the first fundamental form in the usual way by $E = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_u$, $F = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v$ and $G = \boldsymbol{\sigma}_v \cdot \boldsymbol{\sigma}_v$. Consider a torus in $\mathbb{R}^4$, parametrised with**

$$\boldsymbol{\sigma}(u,v) = (a\cos u, a\sin u, b\cos v, b\sin v)\ ,$$

where a and b are positive constants. Show that this torus is locally isometric to the plane.

As $\boldsymbol{\sigma}_u = (-a \sin u, a \cos u, 0, 0)$ and $\boldsymbol{\sigma}_v = (0, 0, -b \sin v, b \cos v)$, the first fundamental form is $a^2 du^2 + b^2 dv^2$. The same first fundamental from is obtained by parametrising the plane by $\boldsymbol{\sigma}(u, v) = (au, bv)$. Therefore the torus is locally isometric to the plane.

4. Compute the principal curvatures of the surface

$$y \cos \frac{z}{a} = x \sin \frac{z}{a}$$

where a is a non-zero constant.
Hint: first parametrise the surface, say with (r, φ) coordinates, so that $z = a\varphi$.

Consider the parametrisation $\boldsymbol{\sigma}(r, \varphi) = (r \cos \varphi, r \sin \varphi, a\varphi)$. Then

$$\begin{aligned}
\boldsymbol{\sigma}_r &= (\cos \varphi, \sin \varphi, 0) \\
\boldsymbol{\sigma}_\varphi &= (-r \sin \varphi, r \cos \varphi, a) \\
\boldsymbol{\sigma}_{rr} &= (0, 0, 0) \\
\boldsymbol{\sigma}_{r,\varphi} &= (-\sin \varphi, \cos \varphi, 0) \\
\boldsymbol{\sigma}_{\varphi,\varphi} &= (-r \cos \varphi, -r \sin \varphi, 0)
\end{aligned}$$

This gives $E = 1$, $F = 0$, $G = r^2 + a^2$, $\sqrt{EG - F^2} = \sqrt{r^2 + a^2}$, $L = 0$, $M = -a/\sqrt{r^2 + a^2}$ and $N = 0$. The principal curvatures are the solutions of

$$\begin{vmatrix} -\kappa & \frac{-a}{\sqrt{r^2+a^2}} \\ \frac{-a}{\sqrt{r^2+a^2}} & -\kappa(r^2 + a^2) \end{vmatrix} = \kappa^2(r^2 + a^2) - \frac{a^2}{r^2 + a^2} = 0$$

so $\kappa = \pm \frac{a}{r^2+a^2}$.

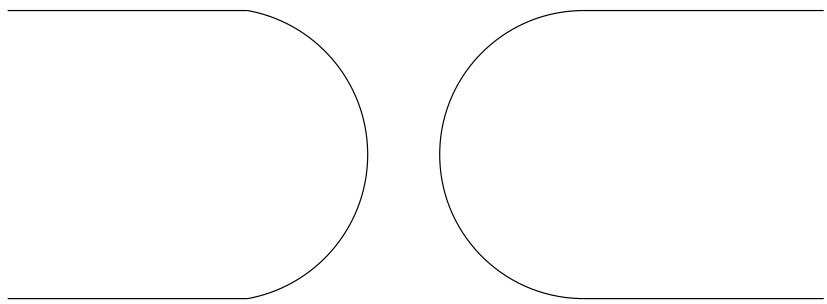
5. a) Show that a geodesic γ on a surface S, which is also a line of curvature, is a plane curve.
b) Show that a geodesic with nowhere vanishing curvature, which lies in a plane, is a line of curvature.
c) Give an example of a line of curvature, which lies in a plane and is not a geodesic.

a) The geodesic condition gives that $\boldsymbol{\gamma}''$ and the surface normal $\mathbf{N}$ are parallel, as the curve normal $\boldsymbol{n}$ is equal to $\mathbf{N}$ or its opposite. On the other hand, $\mathbf{N}'$ and $\boldsymbol{\gamma}'$ are parallel because $\boldsymbol{\gamma}$ is a line of curvature. So $(\mathbf{N} \times \boldsymbol{\gamma}')' = \mathbf{N}' \times \boldsymbol{\gamma}' + \mathbf{N} \times \boldsymbol{\gamma}'' = 0$ and therefore $\mathbf{N} \times \boldsymbol{\gamma}'$ is a constant vector, parallel to the binormal vector of the curve. The curve $\boldsymbol{\gamma}$ lies in a plane with $\mathbf{N} \times \boldsymbol{\gamma}'$ as normal vector, with equation $\boldsymbol{\gamma} \cdot (\mathbf{N} \times \boldsymbol{\gamma}') = \text{const}$, as $(\boldsymbol{\gamma} \cdot (\mathbf{N} \times \boldsymbol{\gamma}'))' = \boldsymbol{\gamma}' \cdot (\mathbf{N} \times \boldsymbol{\gamma}') = 0$.
b) Geodesic implies that $\boldsymbol{\gamma}''$ is a multiple of $\mathbf{N}$. So if $\boldsymbol{\gamma}'' \neq 0$, the vector $\mathbf{N}$ is parallel to the plane of the curve, and the same holds then for $\mathbf{N}'$. Because $\mathbf{N}'$ is perpendicular to $\mathbf{N}$, it has to be a multiple of $\boldsymbol{\gamma}'$, and the curve is a line of curvature.
c) A parallel on a surface of revolution is always a line of curvature, lying in a plane, but it is not a geodesic, unless the tangent plane is parallel to the axis.

6. Describe (qualitatively) the geodesics on the surface of revolution $M \subset \mathbb{R}^3$, given by

$$\begin{aligned}
M = \{(x, y, z) \mid z = 1,\ x^2 + y^2 \geq 1\} \\
\cup \{(x, y, z) \mid z = -1,\ x^2 + y^2 \geq 1\} \\
\cup \{(x, y, z) \mid |z| \leq 1,\ x^2 + y^2 = \varphi(z)\}\ ,
\end{aligned}$$

where $\varphi(1) = \varphi(-1) = 1$ and $\varphi(z)$ is such that M is a smooth surface, with profile curve looking like



The surface consists of two parallel planes with a hole. Some geodesics are easy to see and describe exactly. Every straight line in one of the planes which does not meet the unit circle, is a geodesic. Also every meridian (like the ones drawn) is a geodesic. A parallel is only a geodesic if $\varphi'(z) = 0$; this happens only at the point nearest to the axis.
To describe the remaining geodesics qualitatively, we use Clairaut's Theorem: $r \sin \vartheta = \mathrm{const}$, where $r^2 = x^2 + y^2$. It suffices to look a point on the unit circle, say in the upper plane. The meridian intersects this circle perpendicularly and goes through the hole; the angle ϑ is zero in this case. For an angle close to zero the geodesic still goes through the hole. The tangent to the unit circle is a geodesic, with $\vartheta = 90°$; it just misses the hole. For an angle close to $90°$ the geodesic goes in the hole , but the angle increases until the tangent to the curve is horizontal, and from there the curve comes up again. In between there is an angle ϑ_0 such that the geodesic is trapped in the hole. It has the smallest parallel as limit cycle.