Gauss' Theorema Egregium

The Gaussian curvature K depends only on the first fundamental form.

We claim that we can write

$$(EG-F^2)^2K = \begin{vmatrix} E & F & F_v - \frac{1}{2}G_u \\ F & G & \frac{1}{2}G_v \\ \frac{1}{2}E_u & F_u - \frac{1}{2}E_v & F_{uv} - \frac{1}{2}E_{vv} - \frac{1}{2}G_{uu} \end{vmatrix} - \begin{vmatrix} E & F & \frac{1}{2}E_v \\ F & G & \frac{1}{2}G_u \\ \frac{1}{2}E_v & \frac{1}{2}G_u & 0 \end{vmatrix} .$$

We know that $(EG-F^2)^2K = (EG-F^2)(LN-M^2)$ and that the coefficients L, M and N are given by

$$\begin{aligned} \sqrt{EG-F^2}\, L &= \det(\sigma_u, \sigma_v, \sigma_{uu}) \\ \sqrt{EG-F^2}\, M &= \det(\sigma_u, \sigma_v, \sigma_{uv}) \\ \sqrt{EG-F^2}\, N &= \det(\sigma_u, \sigma_v, \sigma_{vv}) \end{aligned}$$

Therefore

$$(EG-F^2)(LN-M^2) = \det(\sigma_u, \sigma_v, \sigma_{uu}) \det(\sigma_u, \sigma_v, \sigma_{vv}) - (\det(\sigma_u, \sigma_v, \sigma_{uv}))^2 .$$

Let R be a matrix with rows r_i and C a matrix with columns c_j. The entry on place (i,j) of the matrix RC is equal to the scalar product $r_i \cdot c_j$. So

$$\begin{aligned} &\det(\sigma_u, \sigma_v, \sigma_{uu}) \det(\sigma_u, \sigma_v, \sigma_{vv}) - (\det(\sigma_u, \sigma_v, \sigma_{uv}))^2 \\ &= \begin{vmatrix} \sigma_u\cdot\sigma_u & \sigma_u\cdot\sigma_v & \sigma_u\cdot\sigma_{vv} \\ \sigma_v\cdot\sigma_u & \sigma_v\cdot\sigma_v & \sigma_v\cdot\sigma_{vv} \\ \sigma_{uu}\cdot\sigma_u & \sigma_{uu}\cdot\sigma_v & \sigma_{uu}\cdot\sigma_{vv} \end{vmatrix} - \begin{vmatrix} \sigma_u\cdot\sigma_u & \sigma_u\cdot\sigma_v & \sigma_u\cdot\sigma_{uv} \\ \sigma_v\cdot\sigma_u & \sigma_v\cdot\sigma_v & \sigma_v\cdot\sigma_{uv} \\ \sigma_{uv}\cdot\sigma_u & \sigma_{uv}\cdot\sigma_v & \sigma_{uv}\cdot\sigma_{uv} \end{vmatrix} \\ &= \begin{vmatrix} E & F & F_v - \frac{1}{2}G_u \\ F & G & \frac{1}{2}G_v \\ \frac{1}{2}E_u & F_u - \frac{1}{2}E_v & \sigma_{uu}\cdot\sigma_{vv} \end{vmatrix} - \begin{vmatrix} E & F & \frac{1}{2}E_v \\ F & G & \frac{1}{2}G_u \\ \frac{1}{2}E_v & \frac{1}{2}G_u & \sigma_{uv}\cdot\sigma_{uv} \end{vmatrix} . \end{aligned}$$

Here we used the definition of E, F and G, and the fact that $E_u = 2\sigma_u \cdot \sigma_{uu}$, $E_v = 2\sigma_u \cdot \sigma_{uv}$, $F_u = \sigma_{uu}\cdot\sigma_v + \sigma_u\cdot\sigma_{vu}$, $F_v = \sigma_{uv}\cdot\sigma_v + \sigma_u\cdot\sigma_{vv}$, $G_u = 2\sigma_v\cdot\sigma_{vu}$ amd $G_v = 2\sigma_v\cdot\sigma_{vv}$.

Because the minors of position $(3,3)$ in both determinants are equal we can also write

$$\begin{vmatrix} E & F & F_v - \frac{1}{2}G_u \\ F & G & \frac{1}{2}G_v \\ \frac{1}{2}E_u & F_u - \frac{1}{2}E_v & \sigma_{uu}\cdot\sigma_{vv} - \sigma_{uv}\cdot\sigma_{uv} \end{vmatrix} - \begin{vmatrix} E & F & \frac{1}{2}E_v \\ F & G & \frac{1}{2}G_u \\ \frac{1}{2}E_v & \frac{1}{2}G_u & 0 \end{vmatrix} .$$

We derive further:

$$E_{vv} = \frac{\partial}{\partial v}(2\sigma_u\cdot\sigma_{uv}) = 2\sigma_{uv}\cdot\sigma_{uv} + 2\sigma_u\cdot\sigma_{uvv}$$

$$F_{uv} = \frac{\partial}{\partial v}(\sigma_{uu}\cdot\sigma_v + \sigma_u\cdot\sigma_{uv}) = \sigma_{uuv}\cdot\sigma_v + \sigma_{uv}\cdot\sigma_{uv} + \sigma_{uu}\cdot\sigma_{vv} + \sigma_u\cdot\sigma_{uvv}$$

$$G_{uu} = \frac{\partial}{\partial u}(2\sigma_v\cdot\sigma_{uv}) = 2\sigma_{uv}\cdot\sigma_{uv} + 2\sigma_v\cdot\sigma_{uuv}$$

Elimination of $\sigma_u\cdot\sigma_{uvv}$ and $\sigma_v\cdot\sigma_{uuv}$ gives

$$F_{uv} - \tfrac{1}{2}E_{vv} - \tfrac{1}{2}G_{uu} = \sigma_{uu}\cdot\sigma_{vv} - \sigma_{uv}\cdot\sigma_{uv} .$$