Lösningar till tentamensskrivningen
MMG720, Differentialgeometri, 20090307

1. **Prove the Four Vertex Theorem: every convex simple closed curve in $\mathbb{R}^2$ has at least four vertices.**

   If the function κ_s is not constant, it attains its maximum and minimum, say in P och Q. Assume P and Q are the only vertices. The segment PQ, which we may assume to lie on the x-axis, divides the curve in two parts. On one of it $\kappa_s' > 0$, on the other $\kappa_s' < 0$; we may assume that $y < 0$ there. Then $\int_C y\kappa_s' ds > 0$. On the other hand, by partial integration

   $$\int_C y\kappa_s' ds = -\int_C y'\kappa_s ds = \int_C x'' ds = 0 \ ,$$

   because $t' = \kappa_s n$, where $t = \binom{x'}{y'}$ and $n = \binom{-y'}{x'}$. This contradiction shows that κ_s' has one more sign change. If there are three sign changes, then there is a fourth.

2. **Let $\boldsymbol{\gamma}(s)$, $s \in [0, l]$, be a positively oriented closed convex plane curve. Let $l(\boldsymbol{\gamma})$ be its length and $\mathcal{A}(\mathrm{int}(\boldsymbol{\gamma}))$ be the area of its interior. The curve**

   $$\boldsymbol{\delta}(s) = \boldsymbol{\gamma}(s) - r\boldsymbol{n}(s) \ ,$$

   where r is a positive constant, and $\boldsymbol{n}$ is the normal vector, is called a parallel curve to $\boldsymbol{\gamma}$. Show that
   a) $l(\boldsymbol{\delta}) = l(\boldsymbol{\gamma}) + 2\pi r$,
   b) $\mathcal{A}(\mathrm{int}(\boldsymbol{\delta})) = \mathcal{A}(\mathrm{int}(\boldsymbol{\gamma})) + rl(\boldsymbol{\gamma}) + \pi r^2$,
   c) $k_\delta(s) = k_\gamma(s)/(1 + rk_\gamma(s))$, **where k_δ and k_γ are the curvatures of the curves $\boldsymbol{\delta}$ and $\boldsymbol{\gamma}$, respectively.**

   a) We assume that $\boldsymbol{\gamma}$ is unit-speed. We have $\boldsymbol{\delta}' = \boldsymbol{\gamma}' - r\boldsymbol{n}' = \boldsymbol{\gamma}' + rk\boldsymbol{\gamma}' = (1 + rk)\boldsymbol{\gamma}'$, so $l(\boldsymbol{\delta}) = \int_0^l 1 + rk\, ds = l(\boldsymbol{\gamma}) + r\int_0^l k\, ds = l(\boldsymbol{\gamma}) + 2\pi r$.
   b) By Green's theorem $\mathcal{A}(\mathrm{int}(\boldsymbol{\delta})) = \frac{1}{2}\int_0^l \det(\boldsymbol{\delta}, \boldsymbol{\delta}')\, ds$. We compute that $\det(\boldsymbol{\delta}, \boldsymbol{\delta}') = \det(\boldsymbol{\gamma}, \boldsymbol{\gamma}') - r(\det(\boldsymbol{\gamma}, \boldsymbol{n}') + \det(\boldsymbol{n}, \boldsymbol{\gamma}')) - r^2k\det(\boldsymbol{n}, \boldsymbol{\gamma}')$. Observe that $\det(\boldsymbol{n}, \boldsymbol{\gamma}') = -\det(\boldsymbol{\gamma}', \boldsymbol{n}) = -1$, and $\int_0^l \det(\boldsymbol{\gamma}, \boldsymbol{n}')\, ds = \det(\boldsymbol{\gamma}, \boldsymbol{n})|_0^l - \int_0^l \det(\boldsymbol{\gamma}', \boldsymbol{n})\, ds$. As $\boldsymbol{\gamma}$ is a closed curve, $\det(\boldsymbol{\gamma}, \boldsymbol{n})|_0^l = 0$. We are left with the following integral $\frac{1}{2}\int_0^l \det(\boldsymbol{\gamma}, \boldsymbol{\gamma}') + 2r + r^2k\, ds = \mathcal{A}(\mathrm{int}(\boldsymbol{\gamma})) + rl(\boldsymbol{\gamma}) + \pi r^2$.
   c) $k_\delta = \det(\boldsymbol{\delta}', \boldsymbol{\delta}'')/||\boldsymbol{\delta}'||^3$. We know that $\boldsymbol{\delta}' = (1+rk_\gamma)\boldsymbol{\gamma}'$. So $\boldsymbol{\delta}'' = (1+rk_\gamma)\boldsymbol{\gamma}'' + rk_\gamma'\boldsymbol{\gamma}'$ and $\det(\boldsymbol{\delta}', \boldsymbol{\delta}'') = (1 + rk_\gamma)^2\det(\boldsymbol{\gamma}', \boldsymbol{\gamma}'') = k_\gamma(1 + rk_\gamma)^2$. Inserting gives the formula.

3. **Define the tangent space at a point P of a smooth surface S and show that it is a two-dimensional vector space.**

   See Pressley, p. 74.

4. **Show that Scherk's surface**

   $$z = \ln\left(\frac{\cos y}{\cos x}\right)$$

   is a minimal surface, i.e., $H \equiv 0$.

We parametrise and compute derivatives:

$$\begin{aligned}
\boldsymbol{\sigma} &= (u, v, \ln\left(\frac{\cos v}{\cos u}\right)) \\
\boldsymbol{\sigma}_u &= (1, 0, \tan u) \\
\boldsymbol{\sigma}_v &= (0, 1, -\tan v) \\
\boldsymbol{\sigma}_{uu} &= (0, 0, 1/\cos^2 u) \\
\boldsymbol{\sigma}_{uv} &= (0, 0, 0) \\
\boldsymbol{\sigma}_{vv} &= (0, 0, -1/cos^2 v)
\end{aligned}$$

This gives $E = 1 + \tan^2 u = 1/\cos^2 u$, $F = -\tan u \tan v$, $G = 1 + \tan^2 v = 1/\cos^2 v$. Therefore $EG - F^2 = 1 + \tan^2 u + \tan^2 v$. We find $L = 1/\cos^2 u\sqrt{EG - F^2}$, $M = 0$ och $N = -1/\cos^2 v\sqrt{EG - F^2}$. To find the principal curvatures, we have to solve, upon replacing κ by a multiple,

$$\det\begin{pmatrix} (1-\lambda)/\cos^2 u & \lambda \tan u \tan v \\ \lambda \tan u \tan v & (-1-\lambda)/\cos^2 v \end{pmatrix} = 0 \ .$$

This gives $(\lambda^2 - 1)/\cos^2 u \cos^2 v - \lambda^2 \tan^2 u \tan^2 v = 0$. Without explicitly computing the values of λ and κ one sees that there are two solutions with opposite sign, so $H = 0$. Alternatively, one can use the formula $H = \frac{LG - 2MF + NE}{2(EG - F^2)}$. As $M = 0$ it suffices to show that $LG + NE = 0$.

5. Describe the geodesics on a torus.

See Pressley, p. 312.

6. a) Show that a geodesic γ on a surface S, which is also a line of curvature, is a plane curve.

b) Show that a geodesic with nowhere vanishing curvature, which lies in a plane, is a line of curvature.

c) Give an example of a line of curvature, which lies in a plane and is not a geodesic.

(5p).

a) If a geodesic $\boldsymbol{\gamma}$ is a line of curvature, then $\mathbf{N} \times \boldsymbol{\gamma}'$ is a constant vector, where $\mathbf{N}$ is the normal of the surface, because $(\mathbf{N} \times \boldsymbol{\gamma}')' = \mathbf{N}' \times \boldsymbol{\gamma}' + \mathbf{N} \times \boldsymbol{\gamma}''$. The geodesic consdition gives that $\boldsymbol{\gamma}''$ and $\mathbf{N}$ are parallel, whereas $\mathbf{N}'$ and $\boldsymbol{\gamma}'$ are parallel because $\boldsymbol{\gamma}$ is a line of curvature. The curve $\boldsymbol{\gamma}$ lies in a plane with $\mathbf{N} \times \boldsymbol{\gamma}'$ as normal vector, with equation $\boldsymbol{\gamma} \cdot (\mathbf{N} \times \boldsymbol{\gamma}') = \text{const}$, as $(\boldsymbol{\gamma} \cdot (\mathbf{N} \times \boldsymbol{\gamma}'))' = \boldsymbol{\gamma}' \cdot (\mathbf{N} \times \boldsymbol{\gamma}') = 0$.
b) Geodesic implies that $\boldsymbol{\gamma}''$ is a multiple of $\mathbf{N}$. So if $\boldsymbol{\gamma}'' \neq 0$, the vector $\mathbf{N}$ is parallel to the plane of the curve, and the same holds then for $\mathbf{N}'$. Because $\mathbf{N}'$ is perpendicular to $\mathbf{N}$, it has to be a multiple of $\boldsymbol{\gamma}'$, and the curve is a line of curvature.
c) A parallel on a surface of revolution is always a line of curvature, lying in a plane, but it is not a geodesic, unless the tangent plane is parallel to the axis.