THE ISOPERIMETRIC INEQUALITY

The problem, which closed plane curve of given length encloses the largest area, was already known in antiquity, with its solution the circle. According to legend, the Phoenician princess Dido founded the city of Carthago on a piece of land obtained from the local king. As she got only as much land as could be encompassed by an oxhide, she let the hide be cut into thin strips, so that she had enough to encircle an entire hill nearby. A satisfactory proof that the solution is a circle was first obtained in the 19th century.

The famous geometer Jacob Steiner give a proof, based on geometric constructions, which starting from a curve, different from a circle, leads to a curve with the same length, but enclosing a strictly larger area. Dirichlet, his colleague in Berlin, tried without success to convince him that this does not suffice as proof, without showing the existence of a solution.

Many proofs are now known. The one below goes back to Hurwitz. The analytic part can be isolated in the following lemma.

Wirtinger's inequality. *Let $f(t)$ be a piecewise smooth, continuous 2π-periodic function with mean value 0, i.e., $\int_0^{2\pi} f(t)dt = 0$. Then*

$$\int_0^{2\pi} (f')^2 dt \geq \int_0^{2\pi} f^2 dt \ ,$$

with equality if and only if $f(t) = a\cos t + b\sin t$, where a and b are constants.

The condition for equality can also be stated as $f(t) = a\cos(t+b)$, where a and b are (new) constants; this follows from the addition formulas for sine and cosine. Using Wirtinger's inequality we can give a simple proof of the isoperimetric inequality.

Theorem. *Let $\boldsymbol{\gamma}$ be a simple closed curve with length l, enclosing an area A. Then $l^2 \geq 4\pi A$ with equality if and only if $\boldsymbol{\gamma}$ is a circle.*

Proof. We parametrise $\boldsymbol{\gamma}$ with constant speed $l/2\pi$. By a translation we may assume that $\int_0^{2\pi} x(t)dt = 0$. Now $l^2/2\pi = \int_0^{2\pi} \|\boldsymbol{\gamma}'(t)\|^2 dt = \int_0^{2\pi} (x')^2 + (y')^2 dt$. We compute the area with Green's theorem to be $A = \iint_{\mathrm{int}(\boldsymbol{\gamma})} dxdy = \int_0^{2\pi} xy'dt$. This gives us

$$l^2 - 4\pi A = 2\pi \int_0^{2\pi} (x')^2 + (y')^2 - 2xy'dt = 2\pi \int_0^{2\pi} (x')^2 - x^2 dt + 2\pi \int_0^{2\pi} (y' - x)^2 dt \geq 0 \ ,$$

where the first integral is non-negative by Wirtinger's inequality for the function $x(t)$, and the integrand of the second is non-negative.

Equality holds if and only if $x(t) = a\cos(t+b)$ and $y' = x$, so $y(t) = a\sin(t+b) + c$. Because $\sqrt{(x')^2 + (y')^2} = l/2\pi$ we find $a = l/2\pi$, so γ is a circle with centre $(0, c)$ and radius $l/2\pi$. □

The maybe most natural proof of Wirtinger's inequality uses Fourier analysis. There are also elementary proofs (this does not mean that they are simple). We follow [Hardy, Littlewood, Polya. *Inequalities*]. To start we analyse the proof for the weaker form of the inequality, which is the version in Pressley's book.

Wirtinger's inequality. *For a continuous function $f\colon [0,\pi] \to \mathbb{R}$, smooth on $(0,\pi)$, such that $\lim_{t\to 0^+} f'(t)$ and $\lim_{t\to\pi^-} f'(t)$ exist, and with $f(0) = f(\pi) = 0$, one has $\int_0^\pi f^2 dt \le \int_0^\pi (f')^2 dt$ with equality if and only if $f(t) = a\sin t$.*

We want to show that $\int_0^\pi (f')^2 - f^2 dt \ge 0$. We can do this if we find a suitable function ψ such that

$$\int_0^\pi (f')^2 - f^2 dt = \int_0^\pi (f' - f\psi)^2 dt\ .$$

We rewrite:

$$I = \int_0^\pi 2ff'\psi - f^2(1+\psi^2) dt = 0\ .$$

Integration by parts gives

$$I = f^2\psi\Big|_0^\pi - \int_0^\pi f^2(\psi' + 1 + \psi^2) dt = 0\ .$$

The differential equation $-\psi' = 1 + \psi^2$ has solution $\psi(t) = -\tan(t + t_0)$. The function ψ is defined everywhere on the interval $(0,\pi)$ if we take $t_0 = \frac{1}{2}\pi$, so

$$\psi(t) = \frac{\cos t}{\sin t}\ .$$

We compute with L'Hôpitals rule

$$\lim_{t\to 0^+} \frac{f^2\cos t}{\sin t} = \lim_{t\to 0^+} \frac{2ff'\cos t - f^2\sin t}{\cos t} = 0\ ,$$

as $\lim_{t\to 0^+} f'(t)$ exists; likewise $\lim_{t\to\pi^-} f^2\psi = 0$. Therefore $I = 0$. Now

$$\int_0^\pi (f')^2 - f^2 dt = \int_0^\pi \left(f' - f\frac{\cos t}{\sin t}\right)^2 dt \ge 0$$

with equality if and only if $f' - f\cos t/\sin t = 0$. This is a differential equation with solution $f(t) = a\sin t$.

Proof of Wirtinger's inequality. We cannot use directly the argument above, as $f/\sin t$ has poles in general. The trick is to first observe that $f(t_0) = f(t_0 + \pi)$ for some $0 \le t_0 < \pi$, because the function $h(t) = f(t) - f(t + \pi)$ has opposite signs for $t = 0$ and $t = \pi$. Put $f(t_0) = f(t_0 + \pi) = c$. Now we apply the previous argument to the function $f(t) - c$ with $\psi(t) = \cos(t - t_0)/\sin(t - t_0)$. This gives

$$\int_0^{2\pi} (f')^2 - (f-c)^2 dt - \int_0^{2\pi} \left(f' - (f-c)\frac{\cos(t-t_0)}{\sin(t-t_0)} \right)^2 dt = (f-c)^2 \frac{\cos(t-t_0)}{\sin(t-t_0)} \bigg|_0^{2\pi} = 0 \ ,$$

so

$$\int_0^{2\pi} (f')^2 - (f-c)^2 dt \ge 0 \ ,$$

with equality if and only if $f(t) - c = a\sin(t - t_0)$. To get rid of the unknown constant c we use the assumption $\int_0^{2\pi} f dt = 0$, so $2c\int_0^{2\pi} f dt = 0$. Then

$$\int_0^{2\pi} (f')^2 - f^2 dt \ge 2\pi c^2 \ge 0 \ ,$$

with equality if and only if $c = 0$ and $f(t) = a\sin(t - t_0)$. □

Fourier expansion

The functions $\sin mt$, $\cos mt$ form an orthogonal basis of the space of 2π-periodic functions: one has $\int_0^{2\pi} \cos mt \cos nt \, dt = \int_0^{2\pi} \sin mt \sin nt \, dt = 0$ if $m \ne n$, while $\int_0^{2\pi} \cos^2 mt \, dt = \int_0^{2\pi} \sin^2 mt \, dt = \pi$ for $m > 0$, and $\int_0^{2\pi} \cos mt \sin nt \, dt = 0$. This can be shown using the addition formulas for sine and cosine:

$$\cos(m+n)t = \cos mt \cos nt - \sin mt \sin nt \ ,$$
$$\cos(m-n)t = \cos mt \cos nt + \sin mt \sin nt \ .$$

The claim follows because $\int_0^{2\pi} \cos kt \, dt = 0$ if $k \ne 0$, and has value 2π för $k = 0$. In the same way one gets $\int_0^{2\pi} \cos mt \sin nt \, dt = 0$ from the formulas for sine.

Every 2π-periodic function $f(t)$ can be expanded as

$$f(t) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt) \ ,$$

where $a_m = \frac{1}{\pi}\int_0^{2\pi} f(t) \cos mt \, dt$ and $b_m = \frac{1}{\pi}\int_0^{2\pi} f(t) \sin mt \, dt$. The question, when the series converges point-wise $f(t)$, is dealt with in a course on Fourier analysis. It is true for continuous piecewise smooth functions, and we write equality. By differentiating the series we find the Fourier expansion of f' to be

$$f'(t) = \sum_{k=1}^{\infty} (-k a_k \sin kt + k b_k \cos kt) \ .$$

Wirtinger's inequality is now easily proved. The assumption $\int_0^{2\pi} f(t)dt = 0$ gives $a_0 = 0$. Furthermore $\int_0^{2\pi} f^2(t)dt = \sum_{k=1}^{\infty} \pi(a_k^2 + b_k^2)$ (integrate term-wise the square of the Fourier series; actually, the formula can be understood from the fact that $\sin mt$, $\cos mt$ are an orthogonal basis). Likewise $\int_0^{2\pi} (f')^2(t)dt = \sum_{k=1}^{\infty} \pi(k^2 a_k^2 + k^2 b_k^2)$. Therefore

$$\int_0^{2\pi} (f')^2 - f^2 dt = \sum_{k=1}^{\infty} \pi(k^2 - 1)(a_k^2 + b_k^2) \geq 0 ,$$

with equality if and only if $a_k = b_k = 0$ for all $k \geq 2$, i.e., $f(t) = a_1 \cos t + b_1 \sin t$.

Exercises

1. Let $\boldsymbol{\gamma}\colon [\alpha, \beta] \to \mathbb{R}^2$ be a curve with length l and end points on the y-axis, forming a simple closed curve together with the line segment $\overline{\boldsymbol{\gamma}(\alpha)\boldsymbol{\gamma}(\beta)}$. Let A be the enclosed area. Show that $l^2 \geq 2\pi A$ with equality if and only if $\boldsymbol{\gamma}$ is a semicircle. Hint: use the weak form of Wirtinger's inequality.

2. Use the previous exercise to prove the isoperimetric inequality for convex curves.