Lösningar till tentamensskrivningen
MMG720, Differentialgeometri, 20130314

1. **Compute the signed curvature for the logarithmic spiral $\boldsymbol{\gamma}(t) = (e^{at}\cos t, e^{at}\sin t)$.**

   We have $\boldsymbol{\gamma}(t) = e^{at}(\cos t, \sin t)$, so $\boldsymbol{\gamma}'(t) = ae^{at}(\cos t, \sin t) + e^{at}(-\sin t, \cos t)$ and $\boldsymbol{\gamma}''(t) = (a^1 - 1)e^{at}(\cos t, \sin t) + 2ae^{at}(-\sin t, \cos t)$. Then $\|\boldsymbol{\gamma}'(t)\| = e^{at}\sqrt{a^2+1}$ and
   $$\kappa_s = \frac{e^{2at}}{e^{3at}(\sqrt{a^2+1})^3}\begin{vmatrix} a & a^2-1 \\ 1 & 2a \end{vmatrix} = \frac{1}{e^{at}\sqrt{a^2+1}} .$$

2. **Prove the Isoperimetric Inequality for simple closed curves; you may assume Wirtinger's Inequality.**

   See the handout.

3. **Let $\boldsymbol{\gamma}(t)$ be a positively oriented regular plane curve and λ a constant. The *parallel curve* $\boldsymbol{\gamma}_\lambda$ of $\boldsymbol{\gamma}$ is defined by**
   $$\boldsymbol{\gamma}_\lambda(t) = \boldsymbol{\gamma}(t) + \lambda \boldsymbol{n}(t) ,$$
   where $\boldsymbol{n}(t)$ is the positive unit normal vector. Show that
   a) $\boldsymbol{\gamma}_\lambda$ is a regular curve, if $\lambda\kappa_s < 1$ for all t,
   b) the signed curvature of $\boldsymbol{\gamma}_\lambda$ is $\kappa_s/(1-\lambda\kappa_s)$; here κ_s is the signed curvature of $\boldsymbol{\gamma}$.

   We assume that $\boldsymbol{\gamma}$ is unit-speed.
   a) We have $\boldsymbol{\gamma}_\lambda' = \boldsymbol{\gamma}' + \lambda\boldsymbol{n}' = \boldsymbol{\gamma}' - \lambda\kappa_s\boldsymbol{\gamma}' = (1-\lambda\kappa_s)\boldsymbol{\gamma}'$, so $\boldsymbol{\gamma}_\lambda$ is a regular curve, if $\lambda\kappa_s < 1$ for all t.
   b) $\boldsymbol{\gamma}_\lambda' = (1-\lambda\kappa_s)\boldsymbol{\gamma}'' + (1-\lambda\kappa_s)'\boldsymbol{\gamma}'$, so the signed curvature is $\det(\boldsymbol{\gamma}_\lambda', \boldsymbol{\gamma}_\lambda'')/\|\boldsymbol{\gamma}_\lambda'\|^3 = (1-\lambda\kappa_s)^2\det(\boldsymbol{\gamma}', \boldsymbol{\gamma}'')/(1-\lambda\kappa_s)^3 = \kappa_s/(1-\lambda\kappa_s)$.

4. **Find the principal curvatures and the principal vectors associated to each principal curvature for the Enneper surface, parametrised by**
   $$\boldsymbol{\sigma}(u,v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2v, u^2 - v^2) .$$
   We compute derivatives:
   $$\begin{aligned}
   \boldsymbol{\sigma} &= (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2v, u^2 - v^2) \\
   \boldsymbol{\sigma}_u &= (1 - u^2 + v^2, 2uv, 2u) \\
   \boldsymbol{\sigma}_v &= (2uv, 1 + u^2 - v^2, -2v) \\
   \boldsymbol{\sigma}_{uu} &= (-2u, 2v, 2) \\
   \boldsymbol{\sigma}_{uv} &= (2v, 2u, 0) \\
   \boldsymbol{\sigma}_{vv} &= (2u, -2v, -2)
   \end{aligned}$$
   This gives $E = G = (1+u^2+v^2)^2$, $F = 0$. Therefore $\sqrt{EG-F^2} = (1+u^2+v^2)^2$. We find $L = 2(1+u^2+v^2)^2/(1+u^2+v^2)^2 = 2$, $M = 0$ och $N = -2$. The principal curvatures are $L/E = 2/(1+u^2+v^2)^2$ and $N/G = -2/(1+u^2+v^2)^2$ with principal vectors $(1,0)$ and $(0,1)$ respectively.

5. Let σ be the ruled surface

$$\boldsymbol{\sigma}(u,v) = \boldsymbol{\gamma}(u) + v\boldsymbol{\delta}(u) \ ,$$

where γ is a unit-speed curve in $\mathbb{R}^3$ and $\boldsymbol{\delta}(u)$ a unit vector for all u. Prove that the map $\boldsymbol{\sigma} : \mathbb{R}^2 \supset U \to \boldsymbol{\sigma}(U) \subset \mathbb{R}^3$ is an isometry if and only if $\boldsymbol{\delta}(u)$ is constant and γ lies in a plane perpendicular to $\boldsymbol{\delta}$. What kind of surface is σ in this case?

As $\boldsymbol{\sigma}_u = \boldsymbol{\gamma}' + v\boldsymbol{\delta}'$ and $\boldsymbol{\sigma}_v = \boldsymbol{\delta}$ the first fundamental form is $(\boldsymbol{\gamma}' \cdot \boldsymbol{\gamma}' + 2v\boldsymbol{\gamma}' \cdot \boldsymbol{\delta}' + v^2\boldsymbol{\delta}' \cdot \boldsymbol{\delta}')du^2 + 2\boldsymbol{\gamma}' \cdot \boldsymbol{\delta}\, dudv + \boldsymbol{\delta} \cdot \boldsymbol{\delta}\, dv^2 = (1 + 2v\boldsymbol{\gamma}' \cdot \boldsymbol{\delta}' + v^2\boldsymbol{\delta}' \cdot \boldsymbol{\delta}')du^2 + 2\boldsymbol{\gamma}' \cdot \boldsymbol{\delta}\, dudv + dv^2$. Here we used that $\boldsymbol{\delta}$ is a unit vector, so $\boldsymbol{\delta} \cdot \boldsymbol{\delta}' = 0$. The first fundamental form of the plane is $du^2{+}dv^2$. Therefore $\boldsymbol{\sigma}$ is an isometry if and only if $\boldsymbol{\gamma}'{\cdot}\boldsymbol{\delta} = 0$ and $1{+}2v\boldsymbol{\gamma}'{\cdot}\boldsymbol{\delta}'{+}v^2\boldsymbol{\delta}'{\cdot}\boldsymbol{\delta}' = 1$ for all v. The second equation gives the two equations $\boldsymbol{\gamma}' \cdot \boldsymbol{\delta}' = 0$ and $\boldsymbol{\delta}' \cdot \boldsymbol{\delta}' = 0$, so $\boldsymbol{\delta}' = 0$ and $\boldsymbol{\delta}$ is constant. The first equation then gives that $\boldsymbol{\gamma}'$ is perpendicular to $\boldsymbol{\delta}$, so the curve lies in a plane perpendicular to $\boldsymbol{\delta}$. In fact, as $(\boldsymbol{\gamma} \cdot \boldsymbol{\delta})' = 0$, the equation satisfied by $\boldsymbol{\gamma}$ is $\boldsymbol{\gamma} \cdot \boldsymbol{\delta} = k$, k a constant.
The surface is then a generalised cylinder.

6. Describe the geodesics on a spheroid, the surface of revolution obtained by rotating an ellipse around one of its axes.

See Pressley.