Lösningar till tentamensskrivningen
MMG720, Differentialgeometri, 20150603

1. **Compute the curvature and torsion for the curve**

$$\boldsymbol{\gamma}(t) = (3t - t^3, 3t^2, 3t + t^3) \ .$$

We compute

$$\begin{aligned} \boldsymbol{\gamma}'(t) &= (3 - 3t^2, 6t, 3 + 3t^2) \ , \\ \boldsymbol{\gamma}''(t) &= (-6t, 6, 6t) \ , \\ \boldsymbol{\gamma}'''(t) &= (-6, 0, 6) \ . \end{aligned}$$

Then $\|\boldsymbol{\gamma}'\|^2 = 9((1-t^2)^2 + 4t^2 + (1+t^2)^2) = 18(1+t^2)^2$,
$\boldsymbol{\gamma}' \times \boldsymbol{\gamma}'' = 18(t^2 - 1, -2t, 1 + t^2)$,
$\|\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''\|^2 = 2 \cdot (18)^2 \cdot (1+t^2)^2$ and $\det(\boldsymbol{\gamma}', \boldsymbol{\gamma}'', \boldsymbol{\gamma}''') = 3 \cdot 6 \cdot 6 \cdot 2$.
This gives $\kappa = \|\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''\| / \|\boldsymbol{\gamma}'\|^3 = 1/3(1+t^2)^2$ och $\tau = \det(\boldsymbol{\gamma}', \boldsymbol{\gamma}'', \boldsymbol{\gamma}''')/\|\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''\|^2 = 1/3(1+t^2)^2 = \kappa$.

2. **Prove the Four Vertex Theorem: every convex simple closed curve in $\mathbb{R}^2$ has at least four vertices.**

If the function κ_s is not constant, it attains its maximum and minimum, say in P och Q. Assume P and Q are the only vertices. The segment PQ, which we may assume to lie on the x-axis, divides the curve in two parts. On one of it $\kappa_s' > 0$, on the other $\kappa_s' < 0$; we may assume that $y < 0$ there. Then $\int_C y\kappa_s' ds > 0$. On the other hand, by partial integration

$$\int_C y\kappa_s' ds = -\int_C y'\kappa_s ds = \int_C x'' ds = 0 \ ,$$

because $t' = \kappa_s n$, where $t = \binom{x'}{y'}$ and $n = \binom{-y'}{x'}$. This contradiction shows that κ_s' has one more sign change. If there are three sign changes, then there is a fourth.

3. **Define the tangent space at a point P of a smooth surface S and show that it is a two-dimensional vector space.**

See Pressley, p. 74.

4. **Consider the Möbius band with parametrisation**

$$\sigma(t, \vartheta) = ((1 - t\sin\tfrac{\vartheta}{2})\cos\vartheta, (1 - t\sin\tfrac{\vartheta}{2})\sin\vartheta, t\cos\tfrac{\vartheta}{2}) \ ,$$

where $-\frac{1}{2} < t < \frac{1}{2}$ and $0 < \vartheta < 2\pi$. Compute the Gaussian curvature K on the circle given by $t = 0$.

We compute:

$$\begin{aligned}
\sigma_t &= (-\sin\frac{\vartheta}{2}\cos\vartheta, -\sin\frac{\vartheta}{2}\sin\vartheta, \cos\frac{\vartheta}{2}) \\
\sigma_\vartheta &= (-(1-t\sin\frac{\vartheta}{2})\sin\vartheta - \frac{t}{2}\cos\frac{\vartheta}{2}\cos\vartheta, (1-t\sin\frac{\vartheta}{2})\cos\vartheta - \frac{t}{2}\cos\frac{\vartheta}{2}\sin\vartheta, -\frac{t}{2}\sin\frac{\vartheta}{2}) \\
\sigma_\vartheta(0,\vartheta) &= (-\sin\vartheta, \cos\vartheta, 0) \\
\sigma_{tt} &= 0 \\
\sigma_{t\vartheta} &= (\sin\frac{\vartheta}{2}\sin\vartheta - \frac{1}{2}\cos\frac{\vartheta}{2}\cos\vartheta, -\sin\frac{\vartheta}{2}\cos\vartheta - \frac{1}{2}\cos\frac{\vartheta}{2}\sin\vartheta, -\frac{1}{2}\sin\frac{\vartheta}{2}) \\
\sigma_{\vartheta\vartheta}(0,\vartheta) &= (-\cos\vartheta, -\sin\vartheta, 0)
\end{aligned}$$

Therefore $E = 1$, $F = 0$ and $G(0,\vartheta) = 1$. We find $L = 0$, so $K = -M^2/G$. Now $M(0,\vartheta) = \det(\sigma_t, \sigma_\vartheta, \sigma_{t\vartheta})|_{t=0} = \frac{1}{2}$. Therefore $K(0,\vartheta) = -\frac{1}{4}$.

5. Show that Scherk's surface

$$z = \ln\left(\frac{\cos y}{\cos x}\right)$$

is a minimal surface, i.e., $H \equiv 0$.

We parametrise and compute derivatives:

$$\begin{aligned}
\boldsymbol{\sigma} &= (u, v, \ln\left(\frac{\cos v}{\cos u}\right)) \\
\boldsymbol{\sigma}_u &= (1, 0, \tan u) \\
\boldsymbol{\sigma}_v &= (0, 1, -\tan v) \\
\boldsymbol{\sigma}_{uu} &= (0, 0, 1/\cos^2 u) \\
\boldsymbol{\sigma}_{uv} &= (0, 0, 0) \\
\boldsymbol{\sigma}_{vv} &= (0, 0, -1/cos^2 v)
\end{aligned}$$

This gives $E = 1 + \tan^2 u = 1/\cos^2 u$, $F = -\tan u \tan v$, $G = 1 + \tan^2 v = 1/\cos^2 v$. Therefore $EG - F^2 = 1 + \tan^2 u + \tan^2 v$. We find $L = 1/\cos^2 u\sqrt{EG-F^2}$, $M = 0$ och $N = -1/\cos^2 v\sqrt{EG-F^2}$. To find the principal curvatures, we have to solve, upon replacing κ by a multiple,

$$\det\begin{pmatrix} (1-\lambda)/\cos^2 u & \lambda\tan u\tan v \\ \lambda\tan u\tan v & (-1-\lambda)/\cos^2 v \end{pmatrix} = 0\,.$$

This gives $(\lambda^2-1)/\cos^2 u\cos^2 v - \lambda^2\tan^2 u\tan^2 v = 0$. Without explicitly computing the values of λ and κ one sees that there are two solutions with opposite sign, so $H = 0$. Alternatively, one can use the formula $H = \frac{LG-2MF+NE}{2(EG-F^2)}$. As $M = 0$ it suffices to show that $LG + NE = 0$.

6. Describe the geodesics on the circular cone

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid z^2 = a^2(x^2+y^2),\ z > 0\}\,,$$

where $a \in \mathbb{R}_+$. For which values of a do there exist geodesics that intersect themselves? (5p)

The cone is isomatric with a circular sector in the plane. The geodesics on the cone are the images of straight lines in the plane. Half lines through the origin in the plane give half lines through the vertex of the cone. All other geodesics go in both directions to infinity.
So a geodesic γ that intersects itself, goes to infinity in both directions. In the point where γ is closest to the origin, its tangent is horizontal. We may assume that our circular sector in the plane is symmetric with respect to the x-axis and that the closest point to the origin lies on the positive x-axis. Then in the plane the geodesic is a vertical straight line. If it is totally contained in the circular sector, then there is no self-intersection. On the other hand, if the straight line intersects the boundary of the circular section, then the intersection point in the upper half plane and the intersection point in the lower half plane give the same point on the cone, so there is a self intersection.
So there exist self intersections if and only if the circular sector is properly contained in the first and fourth quadrants. A point on the circle with radius 1 on the cone has by Pythagoras' theorem distance $\sqrt{1+a^2}$ to the origin, so the circle corresponds to a circle with radius $\sqrt{1+a^2}$ in the circular sector. Let ϑ be the angle of the circular sector. Then $\vartheta\sqrt{1+a^2} = 2\pi$. The condition for self intersection is hat $\vartheta < \pi$, so $\pi\sqrt{1+a^2} > 2\pi$, yielding $1+a^2 > 4$ so $a > \sqrt{3}$.