Tenta MMGF20/LGMA50 28/8 2018

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1. a. $f'_{v}(a) = \nabla f(a) \cdot v$ $\quad v = \frac{(-1,1)}{\sqrt{2}}$

$\nabla f = \left(2xe^{x^2-y}, -e^{x^2-y}\right)$ $\quad a = (1,2)$

$\nabla f(1,2) = \left(2e^{-1}, -e^{-1}\right)$

$f'_{v}(a) = \left(2e^{-1}, -e^{-1}\right) \cdot \frac{(-1,1)}{\sqrt{2}} =$

$= \frac{e^{-1}}{\sqrt{2}}(-2-1) = \frac{3e^{-1}}{\sqrt{2}}$

b. $f \circ g(x,y) = f(g(x,y)) = f(xy^2) =$

$= \left(x^2y^4, xy^2+1\right)$.

$$(f\circ g)' = \begin{bmatrix} 2xy^4 & 4x^2y^3 \\ y^2 & 2xy \end{bmatrix}$$

Funktionaldeterminanten blir då

$$\begin{vmatrix} 2xy^4 & 4x^2y^3 \\ y^2 & 2xy \end{vmatrix} = 4x^2y^5 - 4x^2y^5 = \underline{\underline{0}}$$

2. Vi vill minimera avståndet mellan origo och (x,y), under bivillkoret att (x,y) ligger i planet $x+3y-2z=4$

Avståndet är $(x^2+y^2+z^2)^{1/2}$, men vi kan lika gärna minimera
$f(x,y,z) = x^2+y^2+z^2$, under bivillkoret
$g(x,y,z) = x+3y-2z = 4$.
Vi ska alltså hitta punkter som uppfyller

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 4 \end{cases}$$

$$\nabla f = (2x, 2y, 2z)$$
$$\nabla g = (1, 3, -2)$$

$$\begin{cases} 2x = \lambda \\ 2y = 3\lambda \\ 2z = -2\lambda \\ x+3y-2z=4 \end{cases}$$

Sätt in $\lambda = 2x$ i (2) och (3):

$$\begin{cases} 2y = 3\cdot 2x \\ 2z = -2\cdot 2x \\ x+3y-2z=4 \end{cases}$$

Sätt in $x = y/3$ i (3) och (4):

$$\begin{cases} z = -2\cdot y/3 \\ \frac{y}{3}+3y-2z=4 \end{cases}$$

Sätt in $z = -\frac{2y}{3}$ i (4):

$$\frac{y}{3}+3y-2\cdot\left(-\frac{2y}{3}\right)=4$$

$$\frac{y+9y+4y}{3}=4 \qquad y=\frac{6}{7} \quad z=-\frac{4}{7} \quad x=\frac{2}{7}$$

Minsta avståndet blir då $\left(\frac{6^2+4^2+2^2}{7^2}\right)^{1/2} =$

$$= \frac{(36+16+4)^{1/2}}{7} = \underline{\underline{\frac{\sqrt{56}}{7}}}$$

3.) a.) Normalen till ytan $z = f(x,y)$ ges av ∇F, där $F(x,y,z) = z - f(x,y) =$
$= z - x^2 - y^2$. $\nabla F = (-2x, -2y, 1)$.

Normalen till $x+y+z=0$ är $(1,1,1)$.

Om $(-2x, -2y, 1) = (1,1,1)$ måste vi ha $x = y = -\frac{1}{2}$, så punkten är $\underline{\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)}$. Tangentplanet blir då

$$(1,1,1)\cdot\left(x+\frac{1}{2}, y+\frac{1}{2}, z-\frac{1}{2}\right) = 0$$

$$\underline{\underline{x+y+z = -\frac{1}{2}}}$$

b.) $\begin{cases} z = x^2+y^2 \\ x+y+z=0 \end{cases}$ Insättning av (1) i (2):
$x+y+x^2+y^2=0$

Kvadratkomplettering ger:

$$\left(x+\frac{1}{2}\right)^2 - \frac{1}{4} + \left(y+\frac{1}{2}\right)^2 - \frac{1}{4} = 0$$

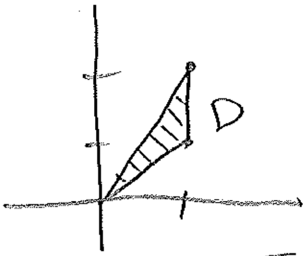
$$\left(x+\frac{1}{2}\right)^2 + \left(y+\frac{1}{2}\right)^2 = \frac{1}{2}$$

Sätt $\begin{cases} x(t) = -\frac{1}{2} + \frac{1}{\sqrt{2}}\cos(t) \\ y(t) = -\frac{1}{2} + \frac{1}{\sqrt{2}}\sin(t) \\ z(t) = x^2+y^2 = \left(-\frac{1}{2}+\frac{1}{\sqrt{2}}\cos(t)\right)^2 + \left(-\frac{1}{2}+\frac{1}{\sqrt{2}}\sin(t)\right)^2 \end{cases}$

$$= \frac{1}{4} - \frac{1}{\sqrt{2}}\cos(t) + \frac{1}{2}\cos^2(t) + \frac{1}{4} - \frac{1}{\sqrt{2}}\sin(t) + \frac{1}{2}\sin^2(t)$$

$$= 1 - \frac{1}{\sqrt{2}}\left(\cos(t)+\sin(t)\right)$$

4. Integranden är inte definierad i origo.



$$\iint_D \frac{1}{x\sqrt{y}}\,dx\,dy = \lim_{\varepsilon\to 0}\int_{\varepsilon}^{1}\frac{1}{x}\int_{x}^{2x}\frac{1}{\sqrt{y}}\,dy\,dx =$$

$$= \lim_{\varepsilon\to 0}\int_{\varepsilon}^{1}\frac{1}{x}\left[2\sqrt{y}\right]_{x}^{2x}dx =$$

$$= \lim_{\varepsilon\to 0}\int_{\varepsilon}^{1}\frac{2}{x}\left(\sqrt{2}\sqrt{x}-\sqrt{x}\right)dx =$$

$$= \lim_{\varepsilon\to 0} 2(\sqrt{2}-1)\int_{\varepsilon}^{1}\frac{1}{\sqrt{x}}\,dx = 2(\sqrt{2}-1)\lim_{\varepsilon\to 0}\left[2\sqrt{x}\right]_{\varepsilon}^{1} =$$

$$= 4(\sqrt{2}-1)\underbrace{\lim_{\varepsilon\to 0}(1-\sqrt{\varepsilon})}_{=1} = \underline{\underline{4(\sqrt{2}-1)}}$$

Svar: konvergent.

5. Vi byter variabler till

$$\left.\begin{aligned} u &= x \\ v &= z-2y \\ w &= z \end{aligned}\right\}$$ så att de nya gränserna blir $0 \le u \le 1,\ 0 \le v \le 1,\ 0 \le w \le 1.$

Funktionaldeterminanten är $\dfrac{d(u,v,w)}{d(x,y,z)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = -2$

så $\dfrac{d(x,y,z)}{d(u,v,w)} = -\dfrac{1}{2}.$

Dessutom har vi $2y = z - v = w - v,$

så $y = \frac{1}{2}(w-v).$

$$\iiint_K xy\;dx\,dy\,dz = -\frac{1}{2}\int_0^1\int_0^1\int_0^1 u\cdot\frac{w-v}{2}\,du\,dv\,dw$$

$$= -\frac{1}{4}\int_0^1\int_0^1 (w-v)\left[\frac{u^2}{2}\right]_0^1 dv\,dw =$$

$$= -\frac{1}{8}\int_0^1\int_0^1 w - v\, dv\,dw = -\frac{1}{8}\int_0^1 \left[wv - \frac{v^2}{2}\right]_0^1 dw$$

$$= -\frac{1}{8}\int_0^1 w - \frac{1}{2}\, dw = -\frac{1}{8}\left[\frac{w^2}{2} - \frac{w}{2}\right]_0^1 =$$

$$= -\frac{1}{8}\left(\frac{1}{2} - \frac{1}{2}\right) = 0$$

6. $$\text{Arean} = \iint_D 1\, dx\,dy = \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{=1 \text{ i vårt fall}} dx\,dy =$$

$$= \int_0^{2\pi} P\,dx + Q\,dy = (*)$$

$$\begin{cases} x(t) = \cos^3(t) \\ y(t) = \sin^3(t) \end{cases} \qquad \begin{cases} x'(t) = 3\cos^2(t)\cdot(-\sin(t)) \\ y'(t) = 3\sin^2(t)\cdot\cos(t) \end{cases}$$

$$\begin{cases} P(r(t)) = -\frac{1}{2}\sin^3(t) \\ Q(r(t)) = \frac{1}{2}\cos^3(t) \end{cases}$$

$$(*) = \frac{3}{2}\int_0^{2\pi} \cos^2(t)\sin^4(t) + \sin^2(t)\cos^4(t)\, dt =$$

$$= \frac{3}{2}\int_0^{2\pi} \cos^2(t)\sin^2(t)\underbrace{\left(\sin^2(t) + \cos^2(t)\right)}_{=1} dt =$$

$$= \frac{3}{2}\int_0^{2\pi} \frac{1+\cos(2t)}{2}\cdot\frac{1-\cos(2t)}{2}\, dt = \frac{3}{8}\int_0^{2\pi} 1 - \cos^2(2t)\, dt =$$

$$= \frac{3}{8}\int_0^{2\pi} 1 - \frac{1+\cos(4t)}{2}\, dt = \frac{3}{16}\left[t - \frac{\sin(4t)}{4}\right]_0^{2\pi} =$$

$= \frac{3}{16} \cdot 2\pi = \frac{3\pi}{8}$

7. a. Se boken, avsnitt 2.2.

b. Se boken, avsnitt 2.4.

8. Se boken, avsnitt 9.4, sats 4.