Lösningar tenta MMGF20

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1. a) Riktningsderivatan ges av $\nabla f(1,-2)\cdot v'$, där v' är normeringen av $(3,-1)$.

$\nabla f = (e^{xy} + xye^{xy} + 2xy,\ x^2e^{xy} + x^2)$

$\nabla f(1,-2) = (e^{-2} - 2e^{-2} + 4,\ e^{-2} + 1) = (4 - e^{-2},\ 1 + e^{-2})$.

$v' = \frac{1}{\sqrt{3^2+1^2}}(3,-1) = \frac{1}{\sqrt{10}}(3,-1)$

$\nabla f(1,-2)\cdot v' = ((4-e^{-2})\cdot 3 + (1+e^{-2})\cdot(-1))\frac{1}{\sqrt{10}} =$

$= \underline{\frac{1}{\sqrt{10}}(11 - 4e^{-2})}$

b) I riktningen $\nabla f(1,-2) = (4-e^{-2},\ 1+e^{-2})$.

2) Vi letar först efter stationära punkter i områdets inre:

$$\begin{cases} f'_x = 3x^2 - 2y^2 + 1 = 0 \\ f'_y = -4xy = 0 \end{cases}$$

Ekv. (2) ger att $x=0$ eller $y=0$:

$x=0$ ger $-2y^2+1=0 \iff 2y^2=1$

$\iff y = \pm\frac{1}{\sqrt{2}}$, dvs $\boxed{(0, \pm\frac{1}{\sqrt{2}})}$

$y=0$ ger $3x^2+1=0 \iff x^2 = -\frac{1}{3}$, som saknar lösning.

Vi tittar nu på randbitarna:

$\underline{x=1:}$ $f(1,y) = 1-2y^2+1+1 = 3-2y^2 = g_1(y)$

$g_1'(y) = -4y = 0 \Rightarrow y=0$, dvs $\boxed{(1,0)}$

$\underline{x=-1:}$ $f(-1,y) = -1+2y^2-1+1 = 2y^2-1 = g_2(y)$

$g_2'(y) = 4y = 0 \Rightarrow y=0$, dvs $\boxed{(-1,0)}$

$\underline{y=\pm 1:}$ $f(x,\pm 1) = x^3-2x+x+1 = x^3-x+1 =$
$= g_3(x)$

$g_3'(x) = 3x^2-1 = 0 \Leftrightarrow$
$x^2 = \frac{1}{3} \Leftrightarrow x = \pm\frac{1}{\sqrt{3}}$

$\boxed{\begin{array}{c}\left(\frac{1}{\sqrt{3}}, \pm 1\right) \\ \left(-\frac{1}{\sqrt{3}}, \pm 1\right)\end{array}}$

Vi måste också titta på värdena i hörnen: $\boxed{(1,\pm 1) \quad (-1,\pm 1)}$.

Nu jämför vi alla värden:

$f(0,\pm\frac{1}{\sqrt{2}}) = 1$ $\qquad f(1,0) = 1+1+1 = 3$

$f(-1,0) = -1-1+1 = -1$ $\qquad f(1,\pm 1) = 1-2+1+1 = 1$

$f(-1,\pm 1) = -1+2-1+1 = 1$

$f\left(\frac{1}{\sqrt{3}},\pm 1\right) = \frac{1}{3\sqrt{3}} - \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + 1 = \frac{1-6+3}{3\sqrt{3}} + 1 =$
$= 1 - \frac{2}{3\sqrt{3}}$ som ligger mellan 0 och 1.

$f\left(-\frac{1}{\sqrt{3}},\pm 1\right) = -\frac{1}{3\sqrt{3}} + \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}} + 1 = 1 + \frac{2}{3\sqrt{3}}$
som ligger mellan 1 och 2.

Max: $\underline{\underline{f(1,0) = 3}}$

Min: $\underline{\underline{f(-1,0) = -1}}$

3. Svaret på första frågan är ja om $f'_y(1,\pi) \neq 0$, enligt implicita funktionssatsen.

$$f'_y = x\cos(y) + xe^{xy}$$

$$f'_y(1,\pi) = \cos(\pi) + e^{\pi} = e^{\pi} - 1 \neq 0.$$

Nu deriverar vi $f(x, y(x))$ implicit m.a.p. x:

$$\sin(y(x)) + x\cos(y(x))\cdot y'(x) + e^{xy(x)}\left(y(x) + xy'(x)\right) = 0$$

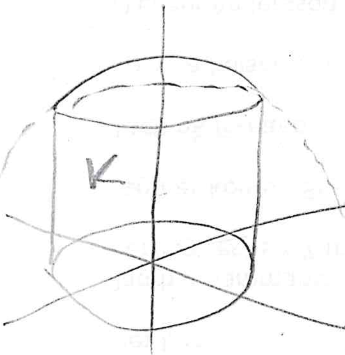
Sätt in $(x,y) = (1,\pi)$:

$$\underbrace{\sin(\pi)}_{=0} + \underbrace{\cos(\pi)}_{=-1}\cdot y'(1) + e^{\pi}\left(\pi + y'(1)\right) = 0$$

$$y'(1)\left(e^{\pi} - 1\right) = -\pi e^{\pi}$$

$$y'(1) = -\frac{\pi e^{\pi}}{e^{\pi} - 1} = \underline{\underline{\frac{-\pi}{1 - e^{-\pi}}}}$$

4. Av symmetriskäl är $x_T = y_T = 0$.



$$m = \iiint_K dx\,dy\,dz = \iint_{x^2+y^2 \leq 1} \left(\int_0^{\sqrt{4-x^2-y^2}} dz \right) dx\,dy =$$

$$\left\{ \begin{matrix} \text{polära} \\ \text{koord.} \end{matrix} \right\} = \int_0^{2\pi}\int_0^1 \sqrt{4-r^2}\, r\,dr\,d\theta = 2\pi \left[-\frac{(4-r^2)^{3/2}}{3/2 \cdot 2} \right]_0^1 =$$

$$= -\frac{2\pi}{3}\left(3^{3/2} - 4^{3/2}\right) = -\frac{2\pi}{3}\left(3\sqrt{3} - 8\right) = \frac{2\pi}{3}\left(8 - 3\sqrt{3}\right)$$

$$\iiint_K z \, dx\,dy\,dz = \iint_{x^2+y^2\le 1}\left(\int_0^{\sqrt{4-x^2-y^2}} z \, dz\right) dx\,dy =$$

$$= \iint_{x^2+y^2\le 1} \left[\frac{z^2}{2}\right]_0^{\sqrt{4-x^2-y^2}} dx\,dy = \int_0^{2\pi}\int_0^1 \frac{4-r^2}{2}\, r \, dr\, d\theta$$

$$= \frac{2\pi}{2}\int_0^1 4r - r^3 \, dr = \pi\left[2r^2 - \frac{r^4}{4}\right]_0^1 =$$

$$= \pi\left(2-\frac{1}{4}\right) = \frac{7\pi}{4}.$$

Så $z_T = \dfrac{\frac{7\pi}{4}}{\frac{2\pi}{3}(8-3\sqrt{3})} = \dfrac{7}{4}\cdot\dfrac{3}{2}\cdot\dfrac{1}{8-3\sqrt{3}} = \underline{\underline{\dfrac{21}{8(8-3\sqrt{3})}}}$

5) $\mathbf{r}'(t) = (-1, 1, 2t)$, så $\int_\gamma \mathbf{F}\cdot d\mathbf{r} =$

$$= \int_0^1 (1-t)\cdot t^2\cdot(-1) + t e^{1-t} + e^{t^2}\cdot 2t \, dt =$$

$$= \int_0^1 t^3 - t^2 + te^{1-t} + 2te^{t^2}\, dt =$$

partial-integreras

$$= \left[\frac{t^4}{4} - \frac{t^3}{3} + e^{t^2}\right]_0^1 + \int_0^1 t e^{1-t}\, dt =$$

$$= \frac{1}{4} - \frac{1}{3} + e^1 - e^0 + \left[-te^{1-t}\right]_0^1 - \int_0^1 -e^{1-t}\, dt =$$

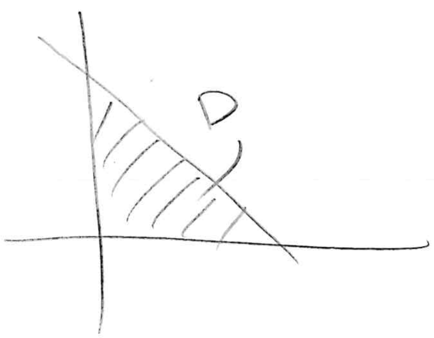
$$= e + \frac{3-4}{12} - 1 - 1 + \left[-e^{1-t}\right]_0^1 =$$

$$= e - \frac{1}{12} - 2 - 1 + e = \underline{\underline{2e - \frac{37}{12}}}$$

6. $$\int_\gamma \underbrace{\left(\ln(1+x^2)+xy^2\right)}_{P}dx + \underbrace{\left(e^{\sin(y)}+xy^2\right)}_{Q}dy = \left\{\begin{matrix}\text{Greens}\\ \text{sats}\end{matrix}\right\}$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_D \left(y^2-2xy\right) dxy = (*)$$

där $D = \{(x,y) : x \geq 0,\ y \geq 0,\ x+y \leq 1\}$.



$$(*) = \int_0^1 \int_0^{1-y} \left(y^2-2xy\right) dx\, dy =$$

$$= \int_0^1 \left[xy^2 - x^2y\right]_0^{1-y} dy =$$

$$= \int_0^1 (1-y)y^2 - (1-y)^2 y\ dy = \int_0^1 y^2-y^3-(1-2y+y^2)y\ dy$$

$$= \int_0^1 -2y^3+3y^2-y\ dy = \left[-\frac{2y^4}{4}+y^3-\frac{y^2}{2}\right]_0^1 =$$

$$= -\frac{1}{2}+1-\frac{1}{2} = \underline{\underline{0}}$$

7. Se boken

8. a) Se boken

b) Se boken

c) $\nabla f(a,b,c)$.