On ergodic and mixing properties of measures on motion groups

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1. mixing by convolutions
2. necessity
3. sufficiency and the spectral radius formula
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- G a locally compact group
- λ_G a left Haar measure on G
- $M(G)$ the space of complex, regular, Borel measures on G
- $\mu \in M(G), \nu \in M(G)$

$$\mu * \nu(A) = \int_G \int_G \mathbf{1}_A(xy) d\mu(x) d\nu(y)$$

$A \subseteq G$, A Borel

- $\mu^n = \mu * \mu * \ldots \mu$ (n-times)

- $\|\mu\|$ the total variation norm on $M(G)$
- $\hat{G}$ the set of equivalence classes of unitary irreducible representations of G, $\mathbf{1}_G$ the trivial representation of G.
- U a unitary representation of G,

$$U(f) = \int_G f(g)U(g)d\lambda_G(g),$$

$f \in L^1(G)$.

- For $\mu \in M(G)$, the Fourier transform of μ is

$$\hat{\mu}(U) = \int_G U(g^{-1})d\mu(g)$$

- If A is a Banach algebra and $a \in A$ we denote by $\rho(a)$ the spectral radius of a.

Definition

$L_0^1(G) = \{f \in L^1(G) : \int_G f(g) d\lambda_G(g) = 0\}$

Definition

A measure $\mu \in M(G)$ is mixing by convolutions if

$$\lim_{n \to \infty} \|f * \mu^n\|_1 = 0$$

for all $f \in L_0^1(G)$.

Theorem (S. Foguel)

Let G be a locally compact abelian group. Let μ be a probability measure on G. Then μ is mixing by convolutions iff

$$|\hat{\mu}(\gamma)| < 1$$

for all $\gamma \in \hat{G} - \{\mathbf{1}_G\}$.

Definition

A locally compact motion group $G = A \rtimes K$ is a semi-direct product of a normal abelian group A and a compact group K.

$AK = KA = G$ and $A \cap K = \{e\}$

Example

G the group of isometries of $\mathbb{R}^n$. Then $G = \mathbb{R}^n \rtimes O(n)$.

Definition

A probability measure $\mu \in M(G)$ is called spread-out if there exists $n \in \mathbb{N}$ such that μ^n is not singular to the Haar measure λ_G of the group G.

Theorem (M. A.-D. Gatzouras)

Let G be a locally compact motion group. Assume that G acts regularly on $\hat{A}$. Let μ be a spread-out probability measure in $M(G)$. Then μ is mixing by convolutions iff

$$\rho(\hat{\mu}(U)) < 1$$

for all $U \in \hat{G} - \{\mathbf{1}_G\}$.

G acts on A by inner automorphisms:

$$i_g(a) = gag^{-1}$$

We obtain an action of G on $\hat{A}$:

$$\tilde{i}_g(\alpha)(a) = \alpha(g^{-1}ag)$$

G acts regularly on $\hat{A}$ if the following conditions are satisfied:

- The orbit space is countably separated
- For each $\alpha \in \hat{A}$, the natural map $gG_\alpha \to \tilde{i}_g(\alpha)$ is a homeomorphism $G/G_\alpha \to \mathcal{O}_\alpha$, where G_α is the stabilizer of α and $\mathcal{O}_\alpha$ the orbit of α.

- For G abelian, the result follows from work of Foguel.
- For G compact, it follows from work of Kawada-Ito.

Theorem (E. Kaniuth)

Let G be a locally compact group. Assume that G has polynomial growth and $L^1(G)$ is symmetric. Let μ be a central probability measure on G. Then μ is mixing by convolutions iff

$$\|\hat{\mu}(U)\| < 1$$

for all $U \in \hat{G} - \{\mathbf{1}_G\}$.

Example

$D_3 =< a, b > \text{ s.t. } a^3 = b^2 = 1, bab = a^2$
$D_3 = \{1, a, a^2, b, ba, ba^2\}.$

$$U_0(a) = U_0(b) = 1$$

$$U_1(a) = 1, U_1(b) = -1$$

$$U_2(a) = \begin{pmatrix} w & 0 \\ 0 & w^2 \end{pmatrix}, U_2(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $w^3 = 1$.

$$\hat{G} = \{U_0, U_1, U_2\}$$

Example

Take μ s.t. $\mu(a^2) = \mu(b) = \frac{1}{2}$. Then

$$\hat{\mu}(U_0) = 1, \hat{\mu}(U_1) = 0$$

$$\hat{\mu}(U_2) = \frac{1}{2} \begin{pmatrix} w & 1 \\ 1 & w^2 \end{pmatrix}$$

$$\|\hat{\mu}(U_2)\| = 1$$

$$\rho(\hat{\mu}(U_2)) = \frac{1}{2}$$

Definition

A linear operator T on a Banach space X is *quasi-compact* if there exist $n \in \mathbb{N}$ and a compact operator Q on X such that $||T^n - Q|| < 1$.

Lemma (Yosida-Kakutani)

Let T be a quasi-compact operator on a Banach space X such that $\sup_{n\in\mathbb{N}} ||T^n|| < \infty$. Then either $\rho(T) < 1$ or $\{z \in \sigma(T) : |z| = 1\}$ contains only eigenvalues of T.

Lemma

Let G be a locally compact CCR group. If μ is a spread-out probability measure in $M(G)$, then $\hat{\mu}(U)$ is quasi-compact for any $U \in \hat{G}$.

proof

Let $n \in \mathbb{N}$ s.t. $\mu^n \neq (\mu^n)_s$. Write $\mu^n = (\mu^n)_{a.c.} + (\mu^n)_s$.
Then

$$\|\widehat{\mu^n}(U) - \widehat{(\mu^n)_{a.c.}}(U)\| = \|\widehat{(\mu^n)_s}(U)\| \leq \|(\mu^n)_s\| < 1,$$

and $\widehat{(\mu^n)_{a.c.}}(U)$ is compact. □

Lemma

Let G be a locally compact group. Then, for any $U \in \hat{G} - \{\mathbf{1}_G\}$, there exists $h \in H_U$ such that $\{U(f)h : f \in L^1_0(G)\}$ is dense in H_U.

proof If not, $L^1_0(G)$ is in the kernel of U and since $L^1_0(G)$ has co-dimension one in $L^1(G)$, U is one-dimensional.
The only one-dimensional representation of G for which the corresponding representation of $L^1(G)$ has kernel $L^1_0(G)$ is $\mathbf{1}_G$. □

Proposition

Let G be a locally compact group, and let μ be a probability measure in $M(G)$ which is mixing by convolutions. Then:

1. $\hat{\mu}(U)^n \to 0$ *in the strong operator topology, for any* $U \in \hat{G} - \{\mathbf{1}_G\}$.
2. $\rho(\hat{\mu}(U)) < 1$ *for any* $U \in \hat{G} - \{\mathbf{1}_G\}$ *for which* $\hat{\mu}(U)$ *is quasi-compact.*

proof

1. Let $U \in \hat{G} - \{\mathbf{1}_G\}$. It is sufficient to show that

$$\|\hat{\mu}(U)^n U(f)h\| \to 0$$

for $f \in L^1(G)$, $h \in H_U$. Set $g(x) = \Delta_G(x^{-1})f(x^{-1})$, where Δ_G is the modular function of G. Then $g \in L^1_0(G)$, $\hat{g}(U) = U(f)$ and

$$\|\hat{\mu}(U)^n U(f)h\| \leq \|\hat{\mu}(U)^n \hat{g}(U)h\| \leq \|(\widehat{g * \mu^n})(U)h\| \leq \|(g*\mu^n)\|_1 \|h\|$$

2. Let $U \in \hat{G} - \{\mathbf{1}_G\}$ and suppose that $\hat{\mu}(U)$ is quasi-compact. By 1, $\hat{\mu}(U)$ cannot have eigenvalues of modulus one. It then follows from Lemma that $\rho(\hat{\mu}(U)) < 1$.

□

Corollary

Let G be a locally compact CCR group. If μ is a spread-out probability measure in $M(G)$ which is mixing by convolutions, then $\rho\big(\hat{\mu}(U)\big) < 1$ for all $U \in \hat{G} \setminus \{\mathbf{1}_G\}$.

proof (T. Ramsey-Y. Weit)
Let G be a locally compact abelian group and μ be a probability measure on G. Assume that

$$|\hat{\mu}(\gamma)| < 1$$

for all $\gamma \in \hat{G} - \{\mathbf{1}_G\}$.
Set $I_\mu = \{f \in L^1(G) : \|f * \mu^n\|_1 \to 0\}$. Then

$$|\hat{f}(\mathbf{1}_G)| = |\widehat{f * \mu^n}(\mathbf{1}_G)| \leq \|f * \mu^n\|_1$$

and $I_\mu \subseteq L_0^1(G)$.
We show that $I_\mu = L_0^1(G)$.
Since $\{e\}$ is a set of synthesis, it is sufficient to show that:
$\{\gamma \in \hat{G} : \hat{f}(\gamma) = 0, \forall f \in I_\mu\} = \{e\}$.

Let $\gamma \in \hat{G} - \{\mathbf{1}_G\}$.
We will find $f \in I_\mu$ such that $\hat{f}(\gamma) \neq 0$.
Let K be a compact neighbourhood of γ, s.t. $e \notin K$. Let $h \in L^1_0(G)$ be such that:

1. $0 \leq \hat{h} \leq 1$
2. $\hat{h}|K = 1$
3. the support of $\hat{h}$ is a compact set not containing e.

Let $f \in L^1_0(G)$ such that $\hat{f}$ is supported in K and $\hat{f}(\gamma) \neq 0$
Then $f * h = f$ and:

$$\|f * \mu^n\|_1 = \|f * h^n * \mu^n\|_1 \leq \|f\|_1 \|(h * \mu)^n\|_1.$$

By the spectral radius formula in $L^1(G)$,

$$\lim_{n \to \infty} \|(h * \mu)^n\|^{\frac{1}{n}} = \sup\{|\hat{h}(\gamma)\hat{\mu}(\gamma)| : \gamma \in \hat{G}\}$$

Since $\hat{h}$ is compactly supported, the supremum is attained and is < 1.
Hence $\|(h * \mu)^n\|_1 \to 0$ and $f \in I_\mu$. $\square$

Let $G = A \rtimes K$ be a locally compact motion group with G acting regularly on $\hat{A}$. Set

$$I_\mu = \{f \in L^1(G) : \|f * \mu^n\|_1 \to 0\}.$$

Then $I_\mu \subseteq L^1_0(G)$. We need to show that $I_\mu = L^1_0(G)$

Theorem

Let $G = A \rtimes K$ be a locally compact motion group with G acting regularly on $\hat{A}$. Then if $\mu \in M(G)$, we have

$$\rho(\mu) = \lim_{n\to\infty} \|\mu^n\|^{\frac{1}{n}} = \sup_{U \in \hat{G}} \rho(\hat{\mu}(U)) \vee \inf_{n \in \mathbb{N}} \|(\mu^n)_s\|^{\frac{1}{n}}$$

Let $\alpha \in \hat{A}$. Set

$$\Lambda_\alpha = \mathrm{ind}_A^G \alpha.$$

Let $\hat{\mathcal{C}}_0$ denote the collection of all compact subsets of $\hat{A}$ not containing $0 \in \hat{A}$, and set

$$I = \bigcup_{\hat{C} \in \hat{\mathcal{C}}_0} \{f \in L^1(G) : \hat{f}(\Lambda_\alpha) = 0, \ \forall \alpha \in \hat{A} - \hat{C}\}.$$

Lemma

Let $f \in I$, such that $f(\Lambda_\alpha) = 0, \forall\, \alpha \in \hat{A} - \hat{C}$, for some compact set $\hat{C} \subseteq \hat{A}$, not containing 0. There exists a central measure ν on G such that:

- $\nu * f = f * \nu = f$
- $\hat{\nu}(\Lambda_\alpha) = 0$, *outside some compact set not containing* 0.

Lemma

Let μ be a spread-out probability measure such that $\rho(\hat{\mu}(U)) < 1$ for all $U \in \hat{G} - \{1_g\}$. Then $I \subseteq I_\mu$.

Then using a result of J. Ludwig on ``noncommutative spectral synthesis´´ in the space of primitive ideals of $L^1(G)$ we prove:

Lemma

The ideal I is dense in $\ker \Lambda_0$ *and hence* $\ker \Lambda_0 \subseteq I_\mu$.

Finally, to treat the $f \in L^1_0(G)$ which do not belong to $\ker(\Lambda_0)$, we use the Peter-Weyl theorem to decompose $\hat{\mu}(\Lambda_0)$ into a direct sum of finite-dimensional operators, which allows us to use the Jordan normal form for each of these operators.

Example

Let $G = \mathbb{Z}_2$ with counting measure and $\mu = h$ the measure that is defined
$h(0) = 0,\ h(1) = 1$. Then $h^2 = h * h$ satisfies

$$h^2(0) = h * h(0) = h(0)h(0) + h(1)h(1) = 1,$$

$$h^2(1) = h * h(1) = h(0)h(1) + h(1)h(0) = 0.$$

$$h^3 = h$$

And generally:
$h^{2m+1} = h$, $h^{2m} = h^2$.

Example

Let $f \in L_0^1(G)$. Then $f(0) = a, \; f(1) = -a$ for some $a \in \mathbb{C}$. We have

$$f * h(0) = f(0)h(0) + f(1)h(1) = -a$$

$$f * h(1) = f(0)h(1) + f(1)h(0) = a$$

and hence

$$f * h = -f.$$

So

$$\|f * h^n\|_1 = \|(-1)^n f\|_1 \not\to 0$$

and h is not mixing.

Example

However if $n = 2m + 1$,

$$\|\frac{1}{n}\sum_{k=0}^{n-1} f * h^k\|_1 = 0$$

and if $n = 2m$,

$$\|\frac{1}{n}\sum_{k=0}^{n-1} f * h^k\|_1 = \|\frac{1}{n} f\|_1 \to 0.$$

Definition

A probability measure $\mu \in M(G)$ is ergodic by convolutions if

$$\lim_{n\to\infty} \|\frac{1}{n}\sum_{k=0}^{n-1} f * \mu^k\| \to 0$$

for all $f \in L^1_0(G)$.

Theorem (M. A.-D. Gatzouras)

Let G be a locally compact motion group. Assume that G acts regularly on $\hat{A}$. Let μ be a spread-out probability measure on G. Then μ is ergodic by convolutions iff

$$1 \notin \sigma(\hat{\mu}(U))$$

for all $U \in \hat{G} - \{\mathbf{1}_G\}$.

σ denotes the spectrum of an operator.

Example

Let $G = \mathbb{Z}_2$ with the counting measure and $\mu = h$ the measure that is defined: $h(0) = 0, \; h(1) = 1.$

$$U_0(0) = U_0(1) = 1, U_1(0) = 1, U_1(1) = -1$$

$$\hat{G} = \{U_0, U_1\}$$

$$\hat{h}(U_1) = -1$$

$$1 \notin \sigma(\hat{h}(U_1))$$

$$\rho(\hat{h}(U_1)) = 1$$

Definition

A measure $\mu \in M(G)$ is called adapted if its support is not contained in any closed proper subgroup of G.

Definition

A measure $\mu \in M(G)$ is called strictly aperiodic if its support is not contained in any coset of a closed proper normal subgroup of G.

In the above example μ is adapted but not strictly aperiodic.

Theorem

Let G be a locally compact group which is abelian or compact. Let μ be a probability measure on G. Then the following are equivalent:

- *μ is mixing by convolutions*
- *μ is adapted and strictly aperiodic*
- $\rho(\hat{\mu}(U)) < 1$ *for all* $U \in \hat{G} - \{\mathbf{1}_G\}$

Theorem

Let G be a locally compact group which is abelian or compact. Let μ be a probability measure on G. Then the following are equivalent:

- *μ is ergodic by convolutions*
- *μ is adapted.*
- $1 \notin \sigma(\hat{\mu}(U))$ *for all* $U \in \hat{G} - \{\mathbf{1}_G\}$

Theorem (W. Jaworski)

Let G be a locally compact, compactly generated, second countable group of polynomial growth. Let μ be a spread-out probability measure on G. Then the following are equivalent:

- *μ is ergodic by convolutions*
- *μ is adapted.*

Theorem (M. A.-D. Gatzouras)

Let G be a locally compact motion group. Assume that G acts regularly on $\hat{A}$. Let μ be a spread-out probability measure in $M(G)$. Then the following are equivalent:

- *μ is mixing by convolutions*
- *μ is adapted and strictly aperiodic*
- *$\rho(\hat{\mu}(U)) < 1$ for all $U \in \hat{G} - \{\mathbf{1}_G\}$*

Theorem (M. A.-D. Gatzouras)

Let G be a locally compact motion group. Assume that G acts regularly on hatA. Let μ be a spread-out probability measure in $M(G)$. Then the following are equivalent:

- μ *is ergodic by convolutions*
- μ *is adapted.*
- $1 \notin \sigma(\hat{\mu}(U))$ *for all* $U \in \hat{G} - \{\mathbf{1}_G\}$

Can we drop the spread-out assumption?

Theorem

Let G be a locally compact group. Let μ be a probability measure on G. Then if μ is ergodic by convolutions it is adapted.

Theorem

Let G be a locally compact group. Let μ be a probability measure on G. Then if μ is mixing by convolutions it is adapted and strictly aperiodic

Example (J. Rosenblatt)

There exist a group G which is a semi-direct product $G = \mathbb{Z}^2 \rtimes \mathbb{Z}$ and a measure $\mu \in M(G)$ s.t.

- μ is spread-out
- μ is adapted and strictly aperiodic
- μ is not ergodic by convolutions.

Since G is not a motion group our theorem does not apply. However, we show that there exists $U \in \hat{G} - \{\mathbf{1}_G\}$ s.t. $1 \in \sigma(\hat{\mu}(U))$.

question

For which groups the following are equivalent?

- μ *is ergodic by convolutions*
- $1 \notin \sigma(\hat{\mu}(U))$ *for all* $U \in \hat{G} - \{\mathbf{1}_G\}$

question

For which groups the following are equivalent?

- μ *is mixing by convolutions*
- $\rho(\hat{\mu}(U)) < 1$ *for all* $U \in \hat{G} - \{\mathbf{1}_G\}$