Harmonic Operators and ideals of the Fourier algebra

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Gothenburgh 2016

The Fourier algebra $A(G)$ and its dual

Let G be an arbitrary loc. compact 2nd countable group.
Represent G on $L^2(G)$ by $(\lambda_s f)(t) = f(s^{-1}t),\ f \in L^2(G)$.

The von Neumann algebra of G:

$$\mathrm{VN}(G) = \overline{\mathrm{span}\{\lambda_s : s \in G\}}^{w*} \subseteq \mathcal{B}(L^2(G)).$$

The Fourier algebra $A(G)$ [Eymard, 1964] is the predual of $\mathrm{VN}(G)$. It can be identified with the space of all functions $u : G \to \mathbb{C}$ of the form

$$u(s) = (\lambda_s f, g)\,.$$

It is in fact a Banach algebra of (continuous) functions on G equipped with the predual norm.
For G abelian, $A(G) = \{\hat{f} : f \in L^1(\hat{G})\}$.

Multipliers of the Fourier algebra

Define

$$MA(G) = \{\sigma : G \to \mathbb{C} : \sigma A(G) \subseteq A(G)\}.$$

These act on $\mathrm{VN}(G)$ by duality: define $\sigma \cdot T$ by

$$\langle \sigma \cdot T, u \rangle = \langle T, \sigma u \rangle, \ u \in A(G).$$

A multiplier σ is called completely bounded when the map $T \to \sigma \cdot T$ is a completely bounded map on $\mathrm{VN}(G) \subseteq \mathcal{B}(L^2(G))$.
Define

$$M^{\mathrm{cb}}A(G) = \{\sigma \in MA(G) : T \to \sigma{\cdot}T \text{ completely bounded on } \mathrm{VN}(G)\}.$$

J. de Canniere - U. Haagerup (1985),
M. Cowling - U. Haagerup (1989).
(When G is abelian, $M^{\mathrm{cb}}A(G) = \{\hat{\mu} : \mu \in M(\hat{G})\}$.)

Harmonic functionals, Harmonic operators

For μ: (prob.) measure on G, an $f \in L^\infty(G)$ s.t.

$$f(x) = \int_G f(y^{-1}x)d\mu(y) \quad \text{(i.e. } \mu * f = f\text{)}$$

is μ-harmonic.

Chu and Lau: Replace $(L^1(G), L^\infty(G)) \rightsquigarrow (A(G), \mathrm{VN}(G))$.

For $\sigma \in M^{\mathrm{cb}}A(G)$, they define σ-harmonic functionals

$$\mathcal{H}_\sigma = \{T \in \mathrm{VN}(G) : \sigma \cdot T = T\} \qquad \text{(cf: } \mu * f = f\text{).}$$

Neufang and Runde define σ-harmonic operators

$$\widetilde{\mathcal{H}}_\sigma = \{T \in \mathcal{B}(L^2(G)) : \sigma \bullet T = T\}$$

using an action $T \to \sigma \bullet T$ of $M^{\mathrm{cb}}A(G)$ on $\mathcal{B}(L^2(G))$ which extends $T \to \sigma \cdot T$.

Harmonic functionals, Harmonic operators

Chu and Lau define σ-harmonic functionals (on $A(G)$)

$$\mathcal{H}_\sigma = \{T \in \mathrm{VN}(G) : \sigma \cdot T = T\}$$

Neufang and Runde define σ-harmonic operators

$$\widetilde{\mathcal{H}}_\sigma = \{T \in \mathcal{B}(L^2(G)) : \sigma \bullet T = T\}$$

They prove that, when σ is a fn. of +ive type, $\sigma(e) = 1$ (defines a state on $C^*(G)$) then
$\widetilde{\mathcal{H}}_\sigma$ is the von Neumann algebra $(\mathcal{H}_\sigma \cup \mathcal{D})''$.

($\mathcal{D}$: the multiplication masa of $L^\infty(G)$.)

In fact: $\widetilde{\mathcal{H}}_\sigma$ turns out to belong to the class of 'jointly invariant subspaces' (Anoussis – K. – Todorov, 2014).

To define the extended action *(Neufang – Ruan – Spronk)*:

$\mathcal{B}(L^2(G))$ as an $A(G)$-module

Preparation: the predual $T(G)$.

The predual of $\mathcal{B}(L^2(G))$ can be identified with the space of all functions of the form

$$h(x,y) = \sum_i f_i(x) g_i(y)$$

where $f_i, g_i \in L^2(G)$ and $\sum_i \|f_i\|_2 \|g_i\|_2 < \infty$ via the duality

$$\langle T, h \rangle_T := \sum_i (T f_i | \bar{g}_i) , \quad (T \in \mathcal{B}(L^2(G))$$

(identify functions differing on a marginally null set). [1]

[1] i.e. agreeing on $N^c \times N^c$ with N null

$\mathcal{B}(L^2(G))$ as an $M^{cb}A(G)$-module

If $\sigma \in M^{cb}A(G)$ then σ also acts on $T(G)$ via multiplication by $N(\sigma)$ where $N(\sigma)(s,t) = \sigma(ts^{-1}),\ s,t \in G$.

The map $h \to N(\sigma)h$ is a bounded operator on $T(G)$ and its norm is the cb norm of σ [Bożejko - Fendler (1984)].

If $\sigma \in M^{cb}A(G)$ then for all $T \in \mathcal{B}(L^2(G))$ define $\sigma \bullet T \in \mathcal{B}(L^2(G))$ by

$$\langle \sigma \bullet T, h \rangle := \langle T, N(\sigma)h \rangle , \quad h \in T(G)$$

Jointly Harmonic Operators

Given a subset $\Sigma \subseteq M^{\mathrm{cb}}A(G)$, we define jointly harmonic functionals (resp. operators)

$$\mathcal{H}_\Sigma = \{T \in \mathrm{VN}(G) : \sigma \cdot T = T, \ \text{ for all } \sigma \in \Sigma\}$$

$$\widetilde{\mathcal{H}}_\Sigma = \{T \in \mathcal{B}(L^2(G)) : \sigma \bullet T = T, \ \text{ for all } \sigma \in \Sigma\},$$

we have $\widetilde{\mathcal{H}}_\Sigma \cap \mathrm{VN}(G) = \mathcal{H}_\Sigma$.

Theorem

For all $\Sigma \subseteq M^{cb}A(G)$ we have $\widetilde{\mathcal{H}}_\Sigma = \mathrm{Bim}(\mathcal{H}_\Sigma)$.

Here $\mathrm{Bim}(\mathcal{H}_\Sigma) = \overline{\mathrm{span}\{M_f T M_g : T \in \mathcal{H}_\Sigma, f, g \in L^\infty(G)\}}^{w*}$ is the weak* closed $\mathcal{D}$-bimodule generated by $\mathcal{H}_\Sigma$.

$\widetilde{\mathcal{H}}_\Sigma = \mathrm{Bim}(\mathcal{H}_\Sigma)$: Ingredients of proof

• $\mathcal{H}_\Sigma = J_\Sigma^\perp$ where

$J_\Sigma := \overline{\mathrm{span}\{\sigma u - u : \sigma \in \Sigma, u \in A(G)\}}^{A(G)} \lhd A(G)$.

But

Theorem: $\mathrm{Bim}(J_\Sigma^\perp) = (\mathrm{Sat}\, J_\Sigma)^\perp$ where

$\mathrm{Sat}\, J_\Sigma := \overline{\mathrm{span}\{N(u)h : u \in J_\Sigma, h \in T(G)\}}^{T(G)}$. Now

$$(\mathrm{Sat}\, J_\Sigma)^\perp = \{T \in \mathcal{B}(L^2) : \langle T, N(u)h\rangle = 0\ \forall u \in J_\Sigma, h \in T(G)\}$$

$$= \{T \in \mathcal{B}(L^2) : \langle u \bullet T, h\rangle = 0\ \forall u \in J_\Sigma, h \in T(G)\}$$

$$= \{T \in \mathcal{B}(L^2) : u \bullet T = 0\ \forall u \in J_\Sigma\}$$

$$\cdots = \{T \in \mathcal{B}(L^2) : (\sigma - 1) \bullet T = 0\ \forall \sigma \in \Sigma\}$$

$$= \widetilde{\mathcal{H}}_\Sigma.$$

Hence $\widetilde{\mathcal{H}}_\Sigma = \mathrm{Bim}(\mathcal{H}_\Sigma)$. $\square$

The annihilator formula

Consider any closed ideal $J \lhd A(G)$.

Let

$$\mathrm{Sat}(J) = \overline{[N(u)h : u \in J, h \in T(G)]}^{T(G)}$$

$$\mathrm{Bim}(J^\perp) = \overline{\mathrm{span}\{M_f T M_g : T \in J^\perp, f, g \in L^\infty(G)\}}^{w*}$$

$$\begin{array}{ccccccc} A(G) & \supseteq & J & \xrightarrow{\perp} & J^\perp & \subseteq & \mathrm{VN}(G) \\ & & \downarrow & & \downarrow & & \\ T(G) & \supseteq & \mathrm{Sat}(J) & \xrightarrow{\perp} & \mathrm{Bim}(J^\perp) & \subseteq & \mathcal{B}(L^2(G)) \end{array}$$

Theorem (AKT, 2014)

Let $J \subseteq A(G)$ be a closed ideal. Then $(\mathrm{Sat}\, J)^\perp = \mathrm{Bim}(J^\perp)$.

Jointly invariant subspaces revisited

We call a weak* closed subspace $\mathcal{U} \subseteq \mathcal{B}(L^2(G))$ jointly invariant if it is simultaneously invariant under
(a) all $T \to M_f T M_g$, $f, g \in L^\infty(G)$, and
(b) all $\mathrm{Ad}\rho_r : T \to \rho_r T \rho_r^*$, $r \in G$
(where $r \to \rho_r$ is the right regular rep. of G).

Theorem

Let $\mathcal{U} \subseteq \mathcal{B}(L^2(G))$ be a weak closed subspace. The following are equivalent:*

(i) the space $\mathcal{U}$ is jointly invariant;

(ii) there exists a closed ideal $J \subseteq A(G)$ such that $\mathcal{U} = \mathrm{Bim}(J^\perp)$;

(iii) there exists a subset $\Sigma \subseteq M^{cb}A(G)$ such that $\mathcal{U} = \widetilde{\mathcal{H}}_\Sigma$.

Algebraic structure

Given $\Sigma \subseteq M^{cb}A(G)$, write
$E(\Sigma) = \{t \in G : \sigma(t) = 1 \text{ for all } \sigma \in \Sigma\}$.

Proposition

If $E(\Sigma)$ is the coset of a closed subgroup of *G then*

$$\widetilde{\mathcal{H}}_\Sigma = \overline{\mathrm{span}\{M_g\lambda_x : x \in E(\Sigma), g \in L^\infty(G)\}}^{w*}$$

and $\widetilde{\mathcal{H}}_\Sigma$ is a weak- closed TRO.*

(TRO means: $A, B, C \in \widetilde{\mathcal{H}}_\Sigma \Rightarrow AB^*C \in \widetilde{\mathcal{H}}_\Sigma$.) Synthesis!

Theorem

If $E(\Sigma)$ is a closed subgroup of *G then $\widetilde{\mathcal{H}}_\Sigma$ is a (type I) von Neumann algebra and*

$$\widetilde{\mathcal{H}}_\Sigma = (\mathcal{D} \cup \mathcal{H}_\Sigma)'' = (\mathcal{D} \cup \{\lambda_x : x \in E(\Sigma)\})''.$$

Algebraic structure II

Suppose each $\sigma \in \Sigma$ is a fn. of +ive type s.t. $\sigma(e) = 1$.
Equivalently, $\Sigma \subseteq$ state space of $C^*(G)$.

Theorem

If Σ is as above, the space $\widetilde{\mathcal{H}}_\Sigma$ is a (type I) von Neumann algebra and

$$\widetilde{\mathcal{H}}_\Sigma = (\mathcal{D} \cup \mathcal{H}_\Sigma)'' = (\mathcal{D} \cup \{\lambda_x : x \in E(\Sigma)\})''.$$

$\rightsquigarrow$ *Neufang-Runde* for $\Sigma = \{\sigma\}$.

A characterisation

Question [V. Runde]

Suppose $\widetilde{\mathcal{H}}_\Sigma$ is a von Neumann algebra. What can you say about $E(\Sigma)$?

Answer

It is a closed subgroup of G; in fact,
$E(\Sigma) = \{t \in G : \sigma \bullet \lambda_t = \lambda_t \text{ for all } \sigma \in \Sigma\}$.

Indeed, $\widetilde{\mathcal{H}}_\Sigma$ is a von Neumann algebra if and only if $\mathcal{H}_\Sigma$ is a von Neumann subalgebra of $\mathrm{VN}(G)$, if and only if $E(\Sigma)$ is a closed subgroup of G.
... and an analogous characterisation exists for the TRO case.

On the equality $J^\perp = \mathfrak{N}(J)$

For $J \lhd A(G)$, $$\mathfrak{N}(J) = \{T \in \mathrm{VN}(G) : u \cdot T = 0\ \forall u \in J\}$$ $$\widetilde{\mathfrak{N}}(J) = \{T \in \mathcal{B}(L^2) : u \bullet T = 0\ \forall u \in J\}$$

Proposition

For all closed $J \lhd A(G)$, the equality $\mathrm{Bim}(J^\perp) = \widetilde{\mathfrak{N}}(J)$ holds.

So $\mathrm{Bim}(J^\perp) = \widetilde{\mathfrak{N}}(J) = \mathrm{Bim}(\mathfrak{N}(J))$. What if we 'un-Bim'?

The equality $J^\perp = \mathfrak{N}(J)$ holds iff J is 'compactly generated'.

Which groups G have this property for all ideals J?

Sufficient condition: Ditkin's condition at infinity: every $u \in A(G)$ is $u = \lim uv_n$ where $\mathrm{supp}\, v_n$ compact.

This is formally weaker than the approximation property of Haagerup and Kraus. (The latter fails for $SL(3, \mathbb{Z})$.)

Thank you!