On C^*-algebras of exponential Lie groups

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Göteborg, August 2016

- Lecture based on joint work with Daniel Beltita (IMAR)
- Aims:
  - G Lie exponential, look at the noncommutative topological quantities associated to $C^*(G)$: real rank, stable rank
  - Nuclear dimension and quasidiagonality

C^*-algebra of an exponential Lie group-1

- ▲ G exponential Lie group, $\exp_G : \mathfrak{g} \to G$ diffeom.
- $C^*(G)$ type I separable C^*-algebra (nuclear)
- $C^*(G)$ is CCR if and only if G is nilpotent.
- If G is not CCR, then there is an normal subgroup $H \subset G$ such that
  $G/H \simeq S_2$, S_3^σ, $\sigma \neq 0$, or S_4, where

- $S_2 = \mathbb{R} \ltimes \mathbb{R}$, $(t, x) \cdot (t', x') = (t + t', x + e^t x')$
- $S_3^\sigma = \mathbb{R} \ltimes \mathbb{R}^2$, $(t, x) \cdot (t', x') = (t + t', x + A(t)x')$,
  $$A(t) = e^{\sigma t} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$
- $S_4 = \mathbb{R}^2 \ltimes \mathbb{R}^2$, $(s, t, x) \cdot (s', t', x') = (t + t', x + B(s, t)x')$
  $$B(s, t) = e^t \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}$$

(Auslander, Moore (1966))

C^*-algebra of an exponential Lie group-2

- $\hat{G} \simeq \widehat{C^*(G)} \simeq \mathfrak{g}^*/G$ (Kirillov-Bernat-Ludwig-Leptin)
- Finite decomposition series (B. Currey (1992)): There exist closed two-sided ideals

$$\{0\} = \mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_n = \mathcal{A}$$

where

- $\mathcal{J}_j/\mathcal{J}_{j-1}$ separable continuous trace
- irred. repres. of $\mathcal{J}_j/\mathcal{J}_{j-1}$ are infinite-dimensional, $j = 1, \ldots, n-1$.
- $\Sigma_j := \widehat{\mathcal{J}_j/\mathcal{J}_{j-1}}$ locally closed, and homeomorphic to a semi-algebraic subset of $\mathfrak{g}^*$ for $j = 1, \ldots, n-1$; $\dim \Sigma_j < \infty$.
- $\mathcal{A}/\mathcal{J}_{n-1} \simeq \mathcal{C}_0([\mathfrak{g}, \mathfrak{g}]^\perp)$, hence $\Sigma_n := \widehat{\mathcal{A}/\mathcal{J}_{n-1}} \simeq \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$.

Real rank, stable rank

▲ $\mathcal{A}$ be any unital C^*-algebra
• $n \geq 1$;
$\mathcal{L}_n(\mathcal{A}) = \{(a_1, \dots, a_n) \in \mathcal{A}^n \mid \mathcal{A}a_1 + \cdots + \mathcal{A}a_n = \mathcal{A}\}$.
$\mathcal{A}^{\mathrm{sa}} := \{a \in \mathcal{A} \mid a = a^*\}$.
• The stable rank of $\mathcal{A}$ is defined by

$$\mathrm{sr}(\mathcal{A}) := \min\{n \geq 1 \mid \mathcal{L}_n(\mathcal{A}) \text{ dense in } \mathcal{A}^n\}$$

with the usual convention $\min \emptyset = \infty$.
• The real rank of $\mathcal{A}$ is defined by

$$\mathrm{RR}(\mathcal{A}) := \min\{n \geq 0 \mid \mathcal{L}_{n+1}(\mathcal{A}) \cap (\mathcal{A}^{\mathrm{sa}})^{n+1} \text{ dense in } (\mathcal{A}^{\mathrm{sa}})^{n+1}\}.$$

▲ $\mathcal{A}$ non-unital C^*-algebra: its real rank and its stable rank are defined as the real rank, respectively the stable rank, of its unitization.

Some results

- $\mathcal{A}$ separable C^*-algebra with continuous trace, for which all its irreducible representations are infinite-dimensional. If moreover $\dim \widehat{\mathcal{A}} < \infty$, then $\mathrm{RR}(\mathcal{A}) \leq 1$.
- $\mathcal{A}$ be any separable C^*-algebra with an ideal $\mathcal{J}$ that is a continuous trace C^*-algebra with $\dim(\widehat{\mathcal{J}}) < \infty$, and such that all irreducible representations of $\mathcal{J}$ are infinite dimensional. Then $\mathrm{RR}(\mathcal{A}) = \max\{\mathrm{RR}(\mathcal{J}), \mathrm{RR}(\mathcal{A}/\mathcal{J})\}$.

$$\Downarrow$$

$\mathcal{A}$ be any separable C^*-algebra with a family of closed two-sided ideals

$$\{0\} = \mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_n = \mathcal{A}$$

where for each $j = 1, \ldots, n$, $\mathcal{J}_j/\mathcal{J}_{j-1}$ has continuous trace, with its spectrum of finite covering dimension, and all its irreducible representations are infinite dimensional. Then

$$\mathrm{RR}(\mathcal{A}) = \max\{\mathrm{RR}(\mathcal{J}_j/\mathcal{J}_{j-1}) \mid j = 1, \ldots, n\}.$$

Ranks of C^*-algebras of exponential Lie groups

Theorem

$G = (\mathfrak{g}, \cdot)$ *exponential Lie group. Then*

$$\mathrm{RR}(C^*(G)) = \dim(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$$

Theorem

$$\mathrm{sr}(C^*(G)) = \begin{cases} 1 & \text{if } G = (\mathbb{R}, +), \\ 1 + \max\{[\frac{1}{2}\,\mathrm{RR}(C^*(G))], 1\} & \text{otherwise.} \end{cases}$$

- nilpotent (not neces. Lie) case: Archbold, Kaniuth (2012)

Abelianization

- $\mathcal{A}$ be any C^*-algebra, $\mathcal{J}(\mathcal{A})$ closed 2-sided ideal gen. by $\mathrm{span}\{[a,b] \mid a,b \in \mathcal{A}\} \Rightarrow \mathcal{A}/\mathcal{J}(\mathcal{A}) \simeq \mathcal{C}_0(\Gamma_{\mathcal{A}})$, for a locally compact $\Gamma_{\mathcal{A}}$, and

$$0 \to \mathcal{J}(\mathcal{A}) \to \mathcal{A} \to \mathcal{C}_0(\Gamma_{\mathcal{A}}) \to 0.$$

- $\mathrm{RR}(\mathcal{A}) \geq \dim(\Gamma_{\mathcal{A}})$.
- $\mathrm{RR}(\mathcal{A}) = \dim(\Gamma_{\mathcal{A}})$ holds for several C^*-algebras:
    - $\mathcal{A} = C^*(G)$ for an exponential Lie group G, where $\Gamma_{\mathcal{A}} = [\mathfrak{g},\mathfrak{g}]^{\perp}$.
    - $\mathcal{A}$ is any AF-algebra, then $\mathrm{RR}(\mathcal{A}) = 0 = \dim(\Gamma_{\mathcal{A}})$.
    - $\mathcal{A}$ is the C^*-algebra gen. by the Toeplitz ops with cont. symbols on $\mathbb{T}$; then $\Gamma_{\mathcal{A}} = \mathbb{T}$, $\mathcal{J}(\mathcal{A}) = \mathcal{K}(L^2(\mathbb{T}))$, and $\mathrm{RR}(\mathcal{A}) = 1 = \dim(\mathbb{T})$.
- $\mathrm{RR}(\mathcal{A}) = \dim(\Gamma_{\mathcal{A}})$ fails to be true in general: $\mathcal{A}$ is a simple C^*-algebra, then $\mathcal{J}(\mathcal{A}) = \mathcal{A} \Rightarrow \Gamma_{\mathcal{A}} = \emptyset$, and there are simple C^*-algebras with positive real rank

\- If $\mathcal{A}$ is a C^*-algebra of real rank zero, we have that $\dim(\Gamma_{\mathcal{A}}) = 0$.

Projections

- $\mathcal{A}$ C^*-algebra, denote $\mathrm{Gr}(\mathcal{A}) := \{p \in \mathcal{A} \mid p = p^2 = p^*\}$.
- $\mathcal{A}$ C^*-algebra s.t. $\widehat{\mathcal{A}}$ is a Hausdorff space.

No connected component of $\widehat{\mathcal{A}}$ is compact $\Rightarrow \mathrm{Gr}(\mathcal{A}) = \{0\}$.

- For any short exact sequence of C^*-algebras

$$0 \to \mathcal{J} \to \mathcal{A} \overset{Q}{\to} \mathcal{A}/\mathcal{J} \to 0$$

if $\mathrm{Gr}(\mathcal{A}/\mathcal{J}) = \{0\}$, then $\mathrm{Gr}(\mathcal{J}) = \mathrm{Gr}(\mathcal{A})$.

Theorem

G be any exponential Lie group, $\mathcal{A} := C^*(G)$.
$\mathcal{J}_0 \subset \mathcal{A}$ the intersection of kernels of all characters of G extended to 1-dimensional $*$-representations of $\mathcal{A}$

$\Rightarrow$
$$\mathrm{Gr}(\mathcal{A}) = \mathrm{Gr}(\mathcal{J}_0).$$

($\mathcal{J}_0 \subsetneqq \mathcal{A}$ if $\dim G > 0$).

Open coadjoint orbits-1

Nontrivial projections appear as soon as the group has open coadjoint orbits:

• G be any connected Lie group with its Lie algebra $\mathfrak{g}$ and the duality pairing $\langle\cdot,\cdot\rangle\colon \mathfrak{g}^*\times\mathfrak{g}\to\mathbb{R}$. For any basis $\{X_1,\dots,X_m\}$ in $\mathfrak{g}$ define the polynomial function

$$P\colon \mathfrak{g}^*\to\mathbb{R},\quad P(\xi):=\det(\langle\xi,[X_j,X_k]\rangle)_{1\le j,k\le m}.$$

Then:

- If $\xi\in\mathfrak{g}^*$, then the coadjoint orbit $\mathcal{O}_\xi:=\mathrm{Ad}^*_G(G)\xi$ is an open subset of $\mathfrak{g}^*$ if and only if $P(\xi)\neq 0$.
- The set Σ_0 of open coadjoint orbits of G is finite and their union is a Zariski open subset of $\mathfrak{g}^*$ which may be empty. Moreover, $|\Sigma_0|$ is even.

• G exponential with $\Sigma_0\neq\emptyset$. Then $\mathcal{J}_0$ contains a closed ideal $\mathcal{J}_{00}\simeq\bigoplus_{|\Sigma_0|}\mathcal{K}$
$\Rightarrow \mathrm{Gr}(\mathcal{J}_{00})\neq\{0\}$.

Open coadjoint orbits-2

- If G not necessarily simply connected, or exponential, then $C^*(G)$ can be primitive.

Examples:

- complex $ax+b$ group;
- $\mathfrak{s} = \mathfrak{n}^- \dotplus \mathfrak{h} \dotplus \mathfrak{n}^+$ complex simple Lie alg., where $\mathfrak{h}$ Cartan subalgebra; the Borel subalgebra $\mathfrak{g} := \mathfrak{h} \ltimes \mathfrak{n}^+$.
  Then G has a nonempty open coadjoint orbit $\iff \mathfrak{s} \in \{\mathrm{so}(2\ell+1,\mathbb{C}), \mathfrak{sp}(2\ell,\mathbb{C}), \mathrm{so}(2\ell,\mathbb{C})$ with $\ell \in 2\mathbb{N}, E_7, E_8, F_4, G_2\}$ (Kostant).
  - $\tau\colon G \to \mathrm{End}(\mathcal{V})$ be a finite-dim. representation, then $\pi\colon G \ltimes \mathcal{V}^*_{\mathbb{R}} \to \mathcal{B}(L^2(\mathcal{V},\mu))$, $(\pi(g,\xi)\varphi)(v) := \mathrm{e}^{\mathrm{i}\xi(v)}\varphi(\tau(g^{-1})v)$ is a continuous unitary irreducible representation. and $\{[\pi]\}$ is dense in $\widehat{G \ltimes \mathcal{V}^*_{\mathbb{R}}}$.

Nuclear dimension and quasidiagonability

▲ $\mathcal{A}$ (nuclear) C^*-algebra; assume that there is a a net $(F_\lambda, \beta_\lambda, \alpha_\lambda)_{\lambda \in \Lambda}$ where F_λ finite dimensional C^*-algebras, $\beta_\lambda \colon A \to F_\lambda$, $\alpha_\lambda \colon F_\lambda \to \mathcal{A}$ completely positive such that

- $\alpha_\lambda \circ \beta_\lambda(a) \to a$ uniformly on finite subsets of $\mathcal{A}$;
- $\|\beta_\lambda\| \leq 1$;
- For every λ, $F_\lambda = F_\lambda^0 \oplus \cdots \oplus F_\lambda^n$ where F_λ^0 are ideals, and $\alpha_\lambda|_{F_\lambda^j}$ is completely contractive and of order zero.

• Then $\dim_{\mathrm{nuc}} \mathcal{A} \leq n$. (Winter-Zacharias 2010)
• If in add. one requires α_λ contraction: $\mathrm{dr}(\mathcal{A}) \leq n$ (Kirchberg-Winter (2004))

▲ A separable continuous trace C^*-algebra, then $\dim_{\mathrm{nuc}} \mathcal{A} = \dim \widehat{\mathcal{A}}$.

▲ $0 \to \mathcal{J} \to \mathcal{A} \to \mathcal{B} \to 0$, then
$\max(\dim_{\mathrm{nuc}} \mathcal{J}, \dim_{\mathrm{nuc}} \mathcal{B}) \leq \dim_{\mathrm{nuc}} \mathcal{A} \leq \dim_{\mathrm{nuc}} \mathcal{B} + \dim_{\mathrm{nuc}} \mathcal{J} + 1$.

▲ If $\mathrm{dr}\, \mathcal{A} < \infty$ then $\mathcal{A}$ is strongly quasidiagonal

Nuclear dimension and quasidiagonality: exponential Lie groups

- G exponential Lie group, then $\dim_{\mathrm{nuc}} C^*(G) < \infty$.
- G nilpotent Lie group, then $C^*(G)$ is CCR hence strongly quasidiagonal.
- If G exponential Lie group and $C^*(G)$ not CCR, then $C^*(G)$ is not strongly quasidiagonal:

- Let $H = S_2$, S_3^σ or S_4. Then there is a irred. repres. $\pi\colon H \mapsto \mathcal{B}(\mathcal{H})$ such that $\pi(C^*(H))$ contains a Fredholm operator of index $\neq 0$.
- $C^*(G)$ has a quotient of the form $C^*(H)$.

Problem:

- $\mathrm{dr}(G) < \infty$ for G Lie & nilpotent