On $\mathcal{C}^*$-algebras of exponential Lie groups

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Aims:

- $G$ Lie exponential, look at the noncommutative topological quantities associated to $C^*(G)$: real rank, stable rank
- Nuclear dimension and quasidiagonality
$C^*$-algebra of an exponential Lie group-1

$G$ exponential Lie group, $\exp_G : \mathfrak{g} \to G$ diffeom.

- $C^*(G)$ type I separable $C^*$-algebra (nuclear)
- $C^*(G)$ is CCR if and only if $G$ is nilpotent.
- If $G$ is not CCR, then there is a normal subgroup $H \subset G$ such that $G/H \simeq S_2, S_3^\sigma, \sigma \neq 0$, or $S_4$, where

$S_2 = \mathbb{R} \times \mathbb{R}$, $(t, x) \cdot (t', x') = (t + t', x + e^t x')$

$S_3^\sigma = \mathbb{R} \times \mathbb{R}^2$, $(t, x) \cdot (t', x') = (t + t', x + A(t)x')$,

$A(t) = e^{\sigma t} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$

$S_4 = \mathbb{R}^2 \times \mathbb{R}^2$, $(s, t, x) \cdot (s', t', x') = (t + t', x + B(s, t)x')$

$B(s, t) = e^t \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}$

(Auslander, Moore (1966))
$\mathbb{C}^*$-algebra of an exponential Lie group

- $\hat{G} \cong \mathbb{C}^*(G) \cong g^*/G$ (Kirillov-Bernat-Ludwig-Leptin)
- Finite decomposition series (B. Currey (1992)): There exist closed two-sided ideals

$$\{0\} = \mathcal{I}_0 \subseteq \mathcal{I}_1 \subseteq \cdots \subseteq \mathcal{I}_n = \mathcal{A}$$

where
- $\mathcal{I}_j/\mathcal{I}_{j-1}$ separable continuous trace
- irred. repres. of $\mathcal{I}_j/\mathcal{I}_{j-1}$ are infinite-dimensional, $j = 1, \ldots, n-1$.
- $\Sigma_j := \overline{\mathcal{I}_j/\mathcal{I}_{j-1}}$ locally closed, and homeomorphic to a semi-algebraic subset of $g^*$ for $j = 1, \ldots, n-1$; $\dim \Sigma_j < \infty$.
- $\mathcal{A}/\mathcal{I}_{n-1} \cong C_0([g, g]^\perp)$, hence $\Sigma_n := \overline{\mathcal{A}/\mathcal{I}_{n-1}} \cong g/[g, g]$. 

\textbf{Real rank, stable rank}

\textcircled{1} $\mathcal{A}$ be any unital $C^*$-algebra

\begin{itemize}
  \item $n \geq 1$;
  \item $\mathcal{L}_n(\mathcal{A}) = \{(a_1, \ldots, a_n) \in \mathcal{A}^n \mid \mathcal{A}a_1 + \cdots + \mathcal{A}a_n = \mathcal{A}\}$. \\
  \item $\mathcal{A}^{sa} := \{a \in \mathcal{A} \mid a = a^*\}$.
\end{itemize}

\textbullet{} The \textbf{stable rank} of $\mathcal{A}$ is defined by

$$sr(\mathcal{A}) := \min\{n \geq 1 \mid \mathcal{L}_n(\mathcal{A}) \text{ dense in } \mathcal{A}^n\}$$

with the usual convention $\min \emptyset = \infty$.

\textbullet{} The \textbf{real rank} of $\mathcal{A}$ is defined by

$$RR(\mathcal{A}) := \min\{n \geq 0 \mid \mathcal{L}_{n+1}(\mathcal{A}) \cap (\mathcal{A}^{sa})^{n+1} \text{ dense in } (\mathcal{A}^{sa})^{n+1}\}.$$ 

\textcircled{2} $\mathcal{A}$ non-unital $C^*$-algebra: its real rank and its stable rank are defined as the real rank, respectively the stable rank, of its unitization.
Some results

- A separable $C^*$-algebra with continuous trace, for which all its irreducible representations are infinite-dimensional. If moreover $\dim \hat{A} < \infty$, then $\text{RR}(A) \leq 1$.

- $A$ be any separable $C^*$-algebra with an ideal $\mathcal{J}$ that is a continuous trace $C^*$-algebra with $\dim(\hat{\mathcal{J}}) < \infty$, and such that all irreducible representations of $\mathcal{J}$ are infinite dimensional. Then $\text{RR}(A) = \max\{\text{RR}(\mathcal{J}), \text{RR}(A/\mathcal{J})\}$.

↓

$A$ be any separable $C^*$-algebra with a family of closed two-sided ideals

$$\{0\} = \mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_n = A$$

where for each $j = 1, \ldots, n$, $\mathcal{J}_j/\mathcal{J}_{j-1}$ has continuous trace, with its spectrum of finite covering dimension, and all its irreducible representations are infinite dimensional. Then

$$\text{RR}(A) = \max\{\text{RR}(\mathcal{J}_j/\mathcal{J}_{j-1}) \mid j = 1, \ldots, n\}.$$
**Ranks of $C^*$-algebras of exponential Lie groups**

**Theorem**

If $G = (\mathfrak{g}, \cdot)$ is an exponential Lie group, then

$$\text{RR}(C^*(G)) = \dim(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$$

**Theorem**

$$\text{sr}(C^*(G)) = \begin{cases} 1 & \text{if } G = (\mathbb{R}, +), \\ 1 + \max\{[\frac{1}{2} \text{RR}(C^*(G))], 1\} & \text{otherwise.} \end{cases}$$

- nilpotent (not neces. Lie) case: Archbold, Kaniuth (2012)
Abelianization

- \( \mathcal{A} \) be any \( C^* \)-algebra, \( \mathcal{J}(\mathcal{A}) \) closed 2-sided ideal gen. by \( \text{span}\{[a, b] \mid a, b \in \mathcal{A}\} \Rightarrow \mathcal{A}/\mathcal{J}(\mathcal{A}) \cong C_0(\Gamma \mathcal{A}) \), for a locally compact \( \Gamma \mathcal{A} \), and

\[
0 \to \mathcal{J}(\mathcal{A}) \to \mathcal{A} \to C_0(\Gamma \mathcal{A}) \to 0.
\]

- \( \text{RR}(\mathcal{A}) \geq \dim(\Gamma \mathcal{A}) \).
- \( \text{RR}(\mathcal{A}) = \dim(\Gamma \mathcal{A}) \) holds for several \( C^* \)-algebras:
  - \( \mathcal{A} = C^*(G) \) for an exponential Lie group \( G \), where \( \Gamma \mathcal{A} = [g, g]^\perp \).
  - \( \mathcal{A} \) is any AF-algebra, then \( \text{RR}(\mathcal{A}) = 0 = \dim(\Gamma \mathcal{A}) \).
  - \( \mathcal{A} \) is the \( C^* \)-algebra gen. by the Toeplitz ops with cont. symbols on \( \mathbb{T} \); then \( \Gamma \mathcal{A} = \mathbb{T}, \mathcal{J}(\mathcal{A}) = \mathcal{K}(L^2(\mathbb{T})) \), and \( \text{RR}(\mathcal{A}) = 1 = \dim(\mathbb{T}) \).
- \( \text{RR}(\mathcal{A}) = \dim(\Gamma \mathcal{A}) \) fails to be true in general: \( \mathcal{A} \) is a simple \( C^* \)-algebra, then \( \mathcal{J}(\mathcal{A}) = \mathcal{A} \Rightarrow \Gamma \mathcal{A} = \emptyset \), and there are simple \( C^* \)-algebras with positive real rank
  - If \( \mathcal{A} \) is a \( C^* \)-algebra of real rank zero, we have that \( \dim(\Gamma \mathcal{A}) = 0 \).
Projections

- A C*-algebra, denote $\text{Gr}(\mathcal{A}) := \{ p \in \mathcal{A} | p = p^2 = p^* \}$.
- A C*-algebra s.t. $\hat{\mathcal{A}}$ is a Hausdorff space.
  No connected component of $\hat{\mathcal{A}}$ is compact $\Rightarrow \text{Gr}(\mathcal{A}) = \{0\}$.
- For any short exact sequence of C*-algebras

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{A} \overset{Q}{\rightarrow} \mathcal{A}/\mathcal{J} \rightarrow 0$$

if $\text{Gr}(\mathcal{A}/\mathcal{J}) = \{0\}$, then $\text{Gr}(\mathcal{J}) = \text{Gr}(\mathcal{A})$.

**Theorem**

$G$ be any exponential Lie group, $\mathcal{A} := C^*(G)$.
$\mathcal{J}_0 \subset \mathcal{A}$ the intersection of kernels of all characters of $G$ extended to 1-dimensional *-representations of $\mathcal{A}$

$\Rightarrow \quad \text{Gr}(\mathcal{A}) = \text{Gr}(\mathcal{J}_0)$.

($\mathcal{J}_0 \not\subseteq \mathcal{A}$ if $\dim G > 0$).
Nontrivial projections appear as soon as the group has open coadjoint orbits:

- $G$ be any connected Lie group with its Lie algebra $\mathfrak{g}$ and the duality pairing $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$. For any basis $\{X_1, \ldots, X_m\}$ in $\mathfrak{g}$ define the polynomial function
  
  $$P : \mathfrak{g}^* \rightarrow \mathbb{R}, \quad P(\xi) := \det(\langle \xi, [X_j, X_k] \rangle)_{1 \leq j, k \leq m}.$$ 

  Then:
  
  - If $\xi \in \mathfrak{g}^*$, then the coadjoint orbit $O_\xi := \text{Ad}^*_G(G)\xi$ is an open subset of $\mathfrak{g}^*$ if and only if $P(\xi) \neq 0$.
  
  - The set $\Sigma_0$ of open coadjoint orbits of $G$ is finite and their union is a Zariski open subset of $\mathfrak{g}^*$ which may be empty. Moreover, $|\Sigma_0|$ is even.

- $G$ exponential with $\Sigma_0 \neq \emptyset$. Then $\mathcal{J}_0$ contains a closed ideal $\mathcal{J}_{00} \simeq \bigoplus_{|\Sigma_0|} \mathcal{K}$
  
  $\Rightarrow \text{Gr}(\mathcal{J}_{00}) \neq \{0\}$. 


If $G$ not necessarily simply connected, or exponential, then $C^*(G)$ can be primitive.

Examples:

- complex $ax + b$ group;
- $s = n^- \oplus h \oplus n^+$ complex simple Lie alg., where $h$ Cartan subalgebra; the Borel subalgebra $g := h \times n^+$.

Then $G$ has a nonempty open coadjoint orbit

$$\iff s \in \{\text{so}(2\ell + 1, \mathbb{C}), \text{sp}(2\ell, \mathbb{C}), \text{so}(2\ell, \mathbb{C}) \text{ with } \ell \in 2\mathbb{N}, E_7, E_8, F_4, G_2\} \text{ (Kostant).}$$

- $\tau : G \rightarrow \text{End}(\mathcal{V})$ be a finite-dim. representation, then
  $$\pi : G \rtimes \mathcal{V}_R^* \rightarrow \mathcal{B}(L^2(\mathcal{V}, \mu)), (\pi(g, \xi)\varphi)(v) := e^{i\xi(v)}\varphi(\tau(g^{-1})v)$$
  is a continuous unitary irreducible representation. and $\{[\pi]\}$ is dense in $G \rtimes \mathcal{V}_R^*$. 

Open coadjoint orbits-2
Nuclear dimension and quasidiagonality

\( \mathcal{A} \) (nuclear) \( C^* \)-algebra; assume that there is a net \((F_\lambda, \beta_\lambda, \alpha_\lambda)_{\lambda \in \Lambda}\) where \( F_\lambda \) finite dimensional \( C^* \)-algebras, \( \beta_\lambda : A \to F_\lambda, \alpha_\lambda : F_\lambda \to A \) completely positive such that

- \( \alpha_\lambda \circ \beta_\lambda(a) \to a \) uniformly on finite subsets of \( \mathcal{A} \);
- \( \| \beta_\lambda \| \leq 1 \);
- For every \( \lambda \), \( F_\lambda = F^0_\lambda \oplus \cdots \oplus F^n_\lambda \) where \( F^0_\lambda \) are ideals, and \( \alpha_\lambda|_{F^j_\lambda} \) is completely contractive and of order zero.

- Then \( \dim_{\text{nuc}} \mathcal{A} \leq n \). (Winter-Zacharias 2010)
- If in add. one requires \( \alpha_\lambda \) contraction: \( \text{dr}(\mathcal{A}) \leq n \) (Kirchberg-Winter (2004))

\( \mathcal{A} \) separable continuous trace \( C^* \)-algebra, then \( \dim_{\text{nuc}} \mathcal{A} = \dim \hat{\mathcal{A}} \).

\( 0 \to \mathcal{J} \to \mathcal{A} \to \mathcal{B} \to 0 \), then
\[
\max(\dim_{\text{nuc}} \mathcal{J}, \dim_{\text{nuc}} \mathcal{B}) \leq \dim_{\text{nuc}} \mathcal{A} \leq \dim_{\text{nuc}} \mathcal{B} + \dim_{\text{nuc}} \mathcal{J} + 1.
\]

If \( \text{dr} \mathcal{A} < \infty \) then \( \mathcal{A} \) is strongly quasidiagonal.
Nuclear dimension and quasidiagonality: exponential Lie groups

- $G$ exponential Lie group, then $\dim_{\text{nuc}} C^*(G) < \infty$.
- $G$ nilpotent Lie group, then $C^*(G)$ is CCR hence strongly quasidiagonal.
- If $G$ exponential Lie group and $C^*(G)$ not CCR, then $C^*(G)$ is not strongly quasidiagonal:
  - Let $H = S_2, S_3^g$ or $S_4$. Then there is an irreducible representation $\pi : H \rightarrow B(\mathcal{H})$ such that $\pi(C^*(H))$ contains a Fredholm operator of index $\neq 0$.
  - $C^*(G)$ has a quotient of the form $C^*(H)$.

Problem:
- $\text{dr}(G) < \infty$ for $G$ Lie & nilpotent