Classification of locally representable actions of finite dimensional quantum groups on AF C*-algebras

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Finite dimensional quantum groups

A finite dimensional quantum group consists of a finite dimensional C^*-algebra A together with a *-homomorphism Δ, called the co-product, that satisfies the co-associative property $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$ and satisfies that $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are equal to $A \otimes A$. If A is a finite dimensional quantum group, there exists a *-homomorphism $\varepsilon : A \to \mathbb{C}$, called the co-unit, such that $(id \otimes \varepsilon)\Delta = id \otimes 1$ and $(\varepsilon \otimes id)\Delta = 1 \otimes id$. Furthermore, there exists a period two *-anti-automorphism $S : A \to A$, called the antipode, such that $m(id \otimes S)\Delta = \varepsilon(\cdot)1 = m(S \otimes id)\Delta$, where m denotes the multiplication map. One often makes use of the Sweedler notation, writing $\sum a_{(1)} \otimes a_{(2)}$ for $\Delta(a)$ and $\sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$ for $(id \otimes \Delta)\Delta(a) = (\Delta \otimes id)\Delta(a)$.

Given a finite dimensional quantum group A, there exists a unique state h, called the Haar state, satisfying the left invariance property $(id \otimes h)\Delta(a) = h(a)1$ for all $a \in A$. The dual space $\hat{A}$ can be expressed as $\{h(a\cdot) \mid a \in A\}$, where h is the Haar state. The map $A \to \hat{A}$; $a \mapsto \hat{a} = h(a\cdot)$ is called the Fourier transform. We make $\hat{A}$ into a quantum group with multiplication defined by $\hat{a}\hat{b}(x) = \hat{a} \otimes \hat{b}(\Delta(x))$, adjoint defined by $\hat{a}^*(x) = \overline{\hat{a}(S(x)^*)}$, and co-product defined by $\hat{\Delta}(\hat{a})(x \otimes y) = \hat{a}(xy)$.

The K_0 ring of a finite dimensional quantum group

Let A be a finite dimensional quantum group. If M and N are two left modules over A, then $M \otimes_{\mathbb{C}} N$ is a left module over A via the action $x \cdot (v \otimes w) = \sum (x_{(1)} \cdot v) \otimes (x_{(2)} w)$. This gives rise to a ring structure on $K_0(A)$. For this ring structure, the class of the co-unit projection is an identity. Below, we shall describe how this ring structure is compatible with the representation theory of the quantum group.

Definition

Let (A, Δ) be a finite dimensional quantum group. A representation of A on a finite dimensional Hilbert space H is an element $V \in B(H) \otimes A$ such that $(id \otimes \Delta)(V) = V_{(12)}V_{(13)}$. Note, if H is finite dimensional with dimension n, then this reduces to V being a matrix in $M_n(A)$ such that $\Delta(V_{pq}) = \sum_{k=1}^{n} V_{pk} \otimes V_{kq}$ for $p, q = 1, \ldots, n$. The representation is called non-degenerate if V is invertible, and unitary if V is unitary.

Definition

Let (A, Δ) be a finite dimensional quantum group. Given unitary representations U and V of A on finite dimensional Hilbert spaces H_1 and H_2 respectively, an element $x \in B(H_1, H_2)$ is called an intertwiner if $(x \otimes 1)U = V(x \otimes 1)$. Two representations are called unitarily equivalent if they have a unitary intertwiner.

Definition

Let (A, Δ) be a finite dimensional quantum group. Given unitary representations U and V of A on finite dimensional Hilbert spaces H_1 and H_2 respectively, the tensor product of U with V, denoted $U \otimes V$, is the representation on $H_1 \otimes H_2$ given by
$U \otimes V = U_{(13)} V_{(23)}$.

Next, we define a homomorphism $\varphi_{\hat{V}} : A \to B(H)$ and a $K_0(A)$ element associated with a unitary representation of $\hat{A}$ on a finite dimensional Hilbert space H.

Definition

Let (A, Δ) be a finite dimensional quantum group. Given an element $\hat{V} \in M_n(\hat{A})$, define a map $\varphi_V : A \to M_n(\mathbb{C})$ by $\varphi_V(a) = (id \otimes a)(\hat{V})$.

Theorem

*Let (A, Δ) be a finite dimensional quantum group, and let $\hat{V} \in M_n(\hat{A})$. Then the map $\varphi_{\hat{V}} : A \to M_n(\mathbb{C})$ defined above is a unital *-homomorphism if, and only if, $\hat{V}$ is a unitary representation of $(\hat{A}, \hat{\Delta})$.*

Definition

Let (A, Δ) be a finite dimensional quantum group, and let $\hat{V} \in B(H) \otimes \hat{A}$ be a unitary representation on the finite dimensional Hilbert space H. Make H into a left A module via $a \cdot h = \varphi_{\hat{V}}(a)h$. With this action, H is finitely generated and projective. Write $K_0(\hat{V})$ for the class in $K_0(A)$ of H with this action.

Theorem

Let (A, Δ) be a finite dimensional quantum group. Given unitary representations $\hat{U}$ and $\hat{V}$ of $\hat{A}$ on finite dimensional Hilbert spaces, $K_0(\hat{U}) = K_0(\hat{V})$ if, and only if, $\hat{U}$ is unitarily equivalent to $\hat{V}$. Furthermore, for any $x \in K_0(A)^+$, there exists a unitary representation $\hat{U}$ of $\hat{A}$ with $x = K_0(\hat{U})$, and for any two representations $\hat{U}$ and $\hat{V}$ of $\hat{A}$, $K_0(\hat{U}) \cdot K_0(\hat{V}) = K_0(\hat{U} \otimes \hat{V})$.

Actions and smash products with finite dimensional quantum groups

Definition

Let A be a finite dimensional quantum group and let R be an AF C^*-algebra. An action of A on R is a left A module structure on R such that for all $x, y \in R$ and $a \in A$, we have $a \cdot (xy) = m(\Delta(a) \cdot (x \otimes y))$, where m denotes the multiplication map $m : R \otimes R \to R$ and $A \otimes A$ acts on $R \otimes R$ by the tensor product action. We shall call a triple $(R, A, \triangleright)$ where R is a C^*-algebra, A is a finite dimensional quantum group, and $\triangleright$ is an action of A on R a Hopf C^*-dynamical system. Given two Hopf C^*-dynamical systems $(R_1, A, \triangleright_1)$ and $(R_2, A, \triangleright_2)$ with the same quantum group, we define a morphism from $(R_1, A, \triangleright_1)$ to $(R_2, A, \triangleright_2)$ to be a *-homomorphism $\varphi : R_1 \to R_2$ such that $\varphi(a \cdot r) = a \cdot \varphi(r)$ for all $a \in A$ and $r \in R_1$. In agreement with the case of group actions, we shall also call such a map an equivariant *-homomorphism.

Definition

Given an action of a finite dimensional quantum group A on an AF C^*-algebra R, we say that the action is representable (inner) if there exists a *-homomorphism $\gamma : A \to M(R)$, such that for all $a \in A$ and $x \in R$, we have $a \cdot x = \sum \gamma(a_{(1)})x\gamma(S(a_{(2)}))$, where $M(R)$ denotes the multiplier algebra of R and we have used the Sweedler notation in the sum.

Definition

Given an action of a finite dimensional quantum group A on a C^*-algebra R, a covariant representation of (R, A) consists of a Hilbert space H, and *-homomorphisms $\pi : R \to B(H)$ and $\gamma : A \to B(H)$ such that $\pi(a \cdot x) = \sum \gamma(a_{(1)})\pi(x)\gamma(S(a_{(2)}))$ for all $x \in R$ and $a \in A$.

Definition

Given an action of a finite dimensional quantum group A on an AF C^*-algebra R, we make $R \otimes A$ into an algebra by $(x \otimes a)(x' \otimes a') = (x \otimes 1)(\Delta(a) \cdot (x' \otimes a'))$, where $A \otimes A$ acts on $R \otimes A$ via the tensor product of its actions on itself and R, and extending linearly. With this multiplication, we denote $R \otimes A$ by $R \# A$.

Lemma

*If R is a C^*-algebra and A is a finite dimensional quantum group with an inner action on R given by a *-homomorphism $\gamma : A \to M(R)$, then $R\#A \cong R \otimes A$.*

Proof.

In [DDZ] it is shown that then maps $\delta : R\#A \to R \otimes A$ given by $\delta(r\#a) = \sum r\gamma(a_{(1)}) \otimes a_{(2)}$ and $\lambda : R \otimes A \to R\#A$ given by $\lambda(r \otimes a) = \sum r\gamma(S(a_{(1)}))\#a_{(2)}$ are inverse algebra isomorphisms. □

Lemma

*Given two Hopf C^*-dynamical systems $(R_1, A, \triangleright_1)$ and $(R_2, A, \triangleright_2)$ and an equivariant *-homomorphism $\varphi : R_1 \to R_2$, there exists a unique homomorphism, which we shall denote $\tilde{\varphi}$, from $R_1 \# A$ to $R_2 \# A$ such that $\tilde{\varphi}(r \# a) = (\varphi(r)) \# a$ for all $r \in R_1$ and $a \in A$.*

Lemma

Let A be a finite dimensional quantum group acting representably on a finite dimensional C^*-algebra R, let V be a finitely generated projective left A module, and let W be a finitely generated projective left $R \# A$ module. Then $W \otimes_{\mathbb{C}} V$ becomes a left $R \# A$ module via $(r \# a) \cdot (w \otimes v) = \sum ((r \# a_{(1)}) \cdot w) \otimes (a_{(2)} \cdot v)$, and with this module structure, $W \otimes_{\mathbb{C}} V$ is finitely generated and projective. Furthermore, if $V' \cong V$ as a left A module and $W' \cong W$ as a left $R \# A$ module, then $W \otimes_{\mathbb{C}} V \cong W' \otimes_{\mathbb{C}} V'$ as a left $R \# A$ module.

Definition

Let A be a finite dimensional quantum group acting on a C^*-algebra R. We define an operation $K_0(R\#A) \times K_0(A) \to K_0(R\#A)$ as follows. Let V be a finitely generated left A module, and let W be a finitely generated left $R\#A$ module. Set $[W] \cdot [V] = [W \otimes_{\mathbb{C}} V]$, where the action of $R\#A$ on $W \otimes_{\mathbb{C}} V$ is as in the lemma above. Now extend additively. The operation $\cdot$ satisfies $[W] \cdot ([M][N]) = ([W] \cdot [M]) \cdot [N]$, and so defines a right $K_0(A)$ module structure on $K_0(R\#A)$.

Lemma

Suppose that R is a full matrix algebra and that A acts on it representably. Then the isomorphism $R\#A \cong R \otimes A$ from above induces an isomorphism $K_0(R\#A) \cong K_0(R \otimes A) \cong K_0(A)$ of right $K_0(A)$ modules.

Lemma

*Let $(R_1, A, \triangleright_1)$ and $(R_2, A, \triangleright_2)$ be two Hopf C^*-dynamical systems with R_1, R_2, A finite dimensional and the actions representable, and let $\varphi : R_1 \to R_2$ be an equivariant *-homomorphism. Then $K_0(\tilde{\varphi}) : K_0(R_1\#A) \to K_0(R_2\#A)$ is a right $K_0(A)$ module map.*

The following definition gives a K_0 that will play a role similar to that of the class of the unit for AF algebras.

Definition

Given a finite dimensional quantum group A acting on a C^*-algebra R, we may define an operation $\cdot : (R\#A) \otimes R \to R$ by $(r_1\#a) \otimes r_2 \mapsto r_1(a \cdot r_2)$. This operation makes R into a left $R\#A$ module that is finitely generated and projective. We write $\tilde{R}$ for R with this module structure.

For the next theorem, we observe that S respects equivalence of projections. If we extend S to matrix algebras over A by $x \otimes a \mapsto x^T \otimes S(a)$, where T denotes the transpose of a matrix, we see that this extension also respects equivalence of projections. We can thus define a period two automorphism of $K_0(A)$ by $S_*([p]) = [S(p)]$ for $p \in M_n(A)$.

Theorem

Let A be a finite dimensional quantum group acting on a full matrix algebra R, with action coming from a unitary representation $\hat{U}$ of $\hat{A}$. Then, under the identification of $K_0(R\#A)$ with $K_0(A)$ coming from the isomorphism of $R\#A$ with $R \otimes A$ from above, $[\tilde{R}] = nS_*K_0(\hat{U})$, where n is the size of the matrix algebra R.

Definition

Given two actions $\triangleright_1$ and $\triangleright_2$ of a finite dimensional quantum group A on unital AF C^*-algebras R_1 and R_2 respectively, the tensor product action $\triangleright_1 \otimes \triangleright_2$ of A on $R_1 \otimes R_2$ is defined by $a \cdot (x \otimes y) = \sum (a_{(1)} \triangleright_1 x) \otimes (a_{(2)} \triangleright_2 y)$. (cf. [DDZ])

Theorem

Let $\triangleright_1$ and $\triangleright_2$ be two actions of a finite dimensional quantum group A on full matrix algebras R_1 and R_2 respectively, corresponding to unitary representations $\hat{U}_1$ and $\hat{U}_2$ of $\hat{A}$. Then $\triangleright_1 \otimes \triangleright_2$ is conjugate to the inner action defined by $\hat{U}_1 \otimes \hat{U}_2$.

Existence and uniqueness theorems

Theorem

(Existence) Let $(R_1, A, \triangleright_1)$ *and* $(R_2, A, \triangleright_2)$ *be two representable Hopf* C^**-dynamical systems, with* R_1 *and* R_2 *finite dimensional, and with the same finite dimensional quantum group* A *acting, and suppose that* ψ *is an ordered* $K_0(A)$ *module homomorphism from* $K_0(R_1\#A)$ *to* $K_0(R_2\#A)$ *such that* $\psi([\tilde{R}_1]) = [\tilde{R}_2]$*. Then there exists a unital equivariant *-homomorphism* $\tilde{\psi} : R_1 \to R_2$ *such that* $\psi = K_0(\tilde{\psi})$*.*

Theorem

(Uniqueness) Let $(R_1, A, \triangleright_1)$ *and* $(R_2, A, \triangleright_2)$ *be two representable Hopf* C^**-dynamical systems, with* R_1 *and* R_2 *finite dimensional, and with the same finite dimensional quantum group* A *acting, and let* ψ *and* ϕ *be two equivariant homomorphisms from* R_1 *to* R_2*. If* $K_0(\tilde{\phi}) = K_0(\tilde{\psi})$*, as maps from* $K_0(R_1\#A)$ *to* $K_0(R_2\#A)$*, then there exists an equivariant inner automorphism* $\alpha : R_2 \to R_2$ *such that* $\psi = \alpha \circ \phi$*.*

Classification

Definition

Let A be a finite dimensional quantum group and let R be a unital AF C^*-algebra. An action of A on R will be called locally representable if there exists a sequence $\{R_n\}$ of finite dimensional sub-C^*-algebras of R such that $\overline{\cup_n R_n} = R$, $1 \in R_n \subseteq R_{n+1}$ for all n, each R_n is globally invariant under the action of A on R, and the restriction of the action of A to R_n is representable (inner).

In order to work with inductive limits, it is convenient to make our smash products into C^*-algebras, which the following theorem lets us do.

Theorem

Let A be a finite dimensional quantum group acting locally representably on a unital AF C^*-algebra R, and let $\{R_n\}$ be a sequence of finite dimensional sub-C^*-algebras of R satisfying the conditions in the definition of local representability. Define an operation $x \mapsto x^\dagger$ on $R\#A$ by $(r\#a)^\dagger = \sum a^*_{(1)} \cdot r^* \# a^*_{(2)}$ and extending additively. Then, with $\dagger$ as adjoint operation, $R\#A$ is an AF C^*-algebra, each $R_n\#A$ is self adjoint, and $R\#A = \overline{\cup_n(R_n\#A)}$.

The proof of our main theorem follows the familiar pattern of Elliott intertwining arguments.

Theorem

Let $\{(R_n, A, \triangleright_n), \varphi_{nm}\}$ and $\{(R'_n, A, \triangleright'_n), \varphi'_{nm}\}$ be two directed systems of Hopf C^*-dynamical systems where A is a finite dimensional quantum group, each R_n and R'_n is finite dimensional, and each action is inner. Let $(R, A, \triangleright)$ and $(R', A, \triangleright')$ denote the inductive limit Hopf C^*-dynamical systems respectively. Suppose that ψ is an ordered $K_0(A)$ module isomorphism of $K_0(R\#A)$ with $K_0(R'\#A)$ that takes $[\tilde{R}]$ to $[\tilde{R}']$. Then there exists an equivariant isomorphism $\tilde{\psi}$ of $(R, A, \triangleright)$ with $(R', A, \triangleright')$ such that $\psi = K_0(\tilde{\psi})$.

Example: The Kac-Paljutkin algebra

There exists a unique non-commutative, non-co-commutative, eight dimensional quantum group, discovered by Kac and Paljutkin, which we shall denote KP. See [EV] for a discussion of this quantum group. As a C^* algebra, it is isomorphic to $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$. Writing $e_1, e_2, e_3, e_4, e_{11}, e_{22}, e_{12}, e_{21}$ for a system of matrix units, the co-product, Δ, is given by:

$$\Delta(e_1) = e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 + e_4 \otimes e_4$$
$$+(1/2)(e_{11} \otimes e_{11} + e_{22} \otimes e_{22} + e_{12} \otimes e_{12} + e_{21} \otimes e_{21}),$$

$$\Delta(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 + e_3 \otimes e_4 + e_4 \otimes e_3$$
$$+(1/2)(e_{11} \otimes e_{22} + e_{22} \otimes e_{11} + ie_{12} \otimes e_{21} - ie_{21} \otimes e_{12}),$$

$$\Delta(e_3) = e_1 \otimes e_3 + e_2 \otimes e_4 + e_3 \otimes e_1 + e_4 \otimes e_2$$
$$+(1/2)(e_{11} \otimes e_{11} + e_{22} \otimes e_{22} - e_{12} \otimes e_{12} - e_{21} \otimes e_{21}),$$

$$\Delta(e_4) = e_1 \otimes e_4 + e_2 \otimes e_3 + e_3 \otimes e_2 + e_4 \otimes e_1$$
$$+(1/2)(e_{11} \otimes e_{22} + e_{22} \otimes e_{11} - ie_{12} \otimes e_{21} + ie_{21} \otimes e_{12}),$$

$$\Delta(e_{11}) = e_1 \otimes e_{11} + e_2 \otimes e_{22} + e_3 \otimes e_{11} + e_4 \otimes e_{22} + e_{11} \otimes e_1 + e_{22} \otimes e_2$$

$$+ e_{11} \otimes e_3 + e_{22} \otimes e_4,$$

$$\Delta(e_{12}) = e_1 \otimes e_{12} - ie_2 \otimes e_{21} - e_3 \otimes e_{12} + ie_4 \otimes e_{21} + e_{12} \otimes e_1 + ie_{21} \otimes e_2$$

$$- e_{12} \otimes e_3 - ie_{21} \otimes e_4,$$

$$\Delta(e_{21}) = e_1 \otimes e_{21} + ie_2 \otimes e_{12} - e_3 \otimes e_{21} - ie_4 \otimes e_{12} + e_{21} \otimes e_1 - ie_{12} \otimes e_2$$

$$- e_{21} \otimes e_3 + ie_{12} \otimes e_4,$$

and

$$\Delta(e_{22}) = e_1 \otimes e_{22} + e_2 \otimes e_{11} + e_3 \otimes e_{22} + e_4 \otimes e_{11} + e_{22} \otimes e_1 + e_{11} \otimes e_2$$

$$+ e_{22} \otimes e_3 + e_{11} \otimes e_4.$$

The antipode, S, is given by: $S(e_i) = e_i$ for $i = 1, \ldots, 4$ and $S(e_{ij}) = e_{ji}$ for $i, j = 1, 2$.
Since the co-product is non-symmetric, the dual quantum group is non-commutative, and it is in fact isomorphic to KP. The ring structure on $K_0(KP)$ is given by the following multiplication table, where 1 denotes $[e_1]$:

$\cdot$	1	$[e_2]$	$[e_3]$	$[e_4]$	$[e_{11}]$
1	1	$[e_2]$	$[e_3]$	$[e_4]$	$[e_{11}]$
$[e_2]$	$[e_2]$	1	$[e_4]$	$[e_3]$	$[e_{11}]$
$[e_3]$	$[e_3]$	$[e_4]$	1	$[e_2]$	$[e_{11}]$
$[e_4]$	$[e_4]$	$[e_3]$	$[e_2]$	1	$[e_{11}]$
$[e_{11}]$	$[e_{11}]$	$[e_{11}]$	$[e_{11}]$	$[e_{11}]$	$1 + [e_2] + [e_3] + [e_4]$

Notice that $K_0(KP)$ is a commutative ring.

We consider the action of KP on the UHF algebra M_{2^∞} arising from repeatedly tensoring with the representation corresponding to $\pi_{2\times 2}$. More precisely, $R_n = M_{2^n}$, $\varphi_n = \varphi_{U\otimes\cdots\otimes U}$, where U is the representation corresponding to $\pi_{2\times 2}$, and the connecting maps are $x \mapsto x \otimes 1$. From the multiplication table above, we get that the repeated matrix for our inductive system is the matrix A shown below. Considering A^2 makes it easy to see that $K_0(M_{2^\infty} \# KP) \cong \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}[\frac{1}{2}]$.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

The class of the special element is $(0,1)$ with the identification above. It remains to determine the $K_0(KP)$ module structure. Our inductive system is

$$K_0(KP) \xrightarrow{[e_{11}]\times} K_0(KP) \xrightarrow{[e_{11}]\times} K_0(KP) \xrightarrow{[e_{11}]\times} K_0(KP) \longrightarrow \dots$$

Since multiplication by $[e_{11}]$ absorbs multiplication by each $[e_i]$, we see that the elements $[e_i]$ each act as the identity. Multiplication by $[e_{11}]$ gives the map $(a,b) \mapsto (4b,a)$.

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