Classification of locally representable actions of finite dimensional quantum groups on AF C*-algebras

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A finite dimensional quantum group consists of a finite dimensional $C^*$-algebra $A$ together with a *-homomorphism $\Delta$, called the co-product, that satisfies the co-associative property $(\Delta \otimes id)\Delta = (id \otimes \Delta) \Delta$ and satisfies that $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are equal to $A \otimes A$. If $A$ is a finite dimensional quantum group, there exists a *-homomorphism $\varepsilon : A \to \mathbb{C}$, called the co-unit, such that $(id \otimes \varepsilon) \Delta = id \otimes 1$ and $(\varepsilon \otimes id) \Delta = 1 \otimes id$. Furthermore, there exists a period two *-anti-automorphism $S : A \to A$, called the antipode, such that $m(id \otimes S)\Delta = \varepsilon(\cdot)1 = m(S \otimes id)\Delta$, where $m$ denotes the multiplication map. One often makes use of the Sweedler notation, writing $\sum a(1) \otimes a(2)$ for $\Delta(a)$ and $\sum a(1) \otimes a(2) \otimes a(3)$ for $(id \otimes \Delta) \Delta(a) = (\Delta \otimes id) \Delta(a)$. 
Given a finite dimensional quantum group $A$, there exists a unique state $h$, called the Haar state, satisfying the left invariance property $(id \otimes h)\Delta(a) = h(a)1$ for all $a \in A$. The dual space $\hat{A}$ can be expressed as $\{h(a\cdot) \mid a \in A\}$, where $h$ is the Haar state. The map $A \rightarrow \hat{A}; \ a \mapsto \hat{a} = h(a\cdot)$ is called the Fourier transform. We make $\hat{A}$ into a quantum group with multiplication defined by $\hat{a}\hat{b}(x) = \hat{a} \otimes \hat{b}(\Delta(x))$, adjoint defined by $\hat{a}^*(x) = \hat{a}(S(x)^*)$, and co-product defined by $\hat{\Delta}(\hat{a})(x \otimes y) = \hat{a}(xy)$. 
The $K_0$ ring of a finite dimensional quantum group

Let $A$ be a finite dimensional quantum group. If $M$ and $N$ are two left modules over $A$, then $M \otimes_{\mathbb{C}} N$ is a left module over $A$ via the action $x \cdot (v \otimes w) = \sum (x_1 \cdot v) \otimes (x_2 w)$. This gives rise to a ring structure on $K_0(A)$. For this ring structure, the class of the co-unit projection is an identity. Below, we shall describe how this ring structure is compatible with the representation theory of the quantum group.
Definition
Let \((A, \Delta)\) be a finite dimensional quantum group. A representation of \(A\) on a finite dimensional Hilbert space \(H\) is an element \(V \in B(H) \otimes A\) such that \((id \otimes \Delta)(V) = V_{(12)} V_{(13)}\). Note, if \(H\) is finite dimensional with dimension \(n\), then this reduces to \(V\) being a matrix in \(M_n(A)\) such that \(\Delta(V_{pq}) = \sum_{k=1}^{n} V_{pk} \otimes V_{kq}\) for \(p, q = 1, \ldots, n\). The representation is called non-degenerate if \(V\) is invertible, and unitary if \(V\) is unitary.

Definition
Let \((A, \Delta)\) be a finite dimensional quantum group. Given unitary representations \(U\) and \(V\) of \(A\) on finite dimensional Hilbert spaces \(H_1\) and \(H_2\) respectively, an element \(x \in B(H_1, H_2)\) is called an intertwiner if \((x \otimes 1)U = V(x \otimes 1)\). Two representations are called unitarily equivalent if they have a unitary intertwiner.
Definition
Let \((A, \Delta)\) be a finite dimensional quantum group. Given unitary representations \(U\) and \(V\) of \(A\) on finite dimensional Hilbert spaces \(H_1\) and \(H_2\) respectively, the tensor product of \(U\) with \(V\), denoted \(U \otimes V\), is the representation on \(H_1 \otimes H_2\) given by \(U \otimes V = U_{(13)} V_{(23)}\).

Next, we define a homomorphism \(\varphi_\hat{V} : A \rightarrow B(H)\) and a \(K_0(A)\) element associated with a unitary representation of \(\hat{A}\) on a finite dimensional Hilbert space \(H\).
Definition
Let \((A, \Delta)\) be a finite dimensional quantum group. Given an element \(\hat{V} \in M_n(\hat{A})\), define a map \(\varphi_V : A \rightarrow M_n(\mathbb{C})\) by \(\varphi_V(a) = (id \otimes a)(\hat{V})\).

Theorem
Let \((A, \Delta)\) be a finite dimensional quantum group, and let \(\hat{V} \in M_n(\hat{A})\). Then the map \(\varphi_{\hat{V}} : A \rightarrow M_n(\mathbb{C})\) defined above is a unital *-homomorphism if, and only if, \(\hat{V}\) is a unitary representation of \((\hat{A}, \hat{\Delta})\).
Definition
Let \((A, \Delta)\) be a finite dimensional quantum group, and let \(\hat{V} \in B(H) \otimes \hat{A}\) be a unitary representation on the finite dimensional Hilbert space \(H\). Make \(H\) into a left \(A\) module via \(a \cdot h = \varphi_{\hat{V}}(a)h\). With this action, \(H\) is finitely generated and projective. Write \(K_0(\hat{V})\) for the class in \(K_0(A)\) of \(H\) with this action.

Theorem
Let \((A, \Delta)\) be a finite dimensional quantum group. Given unitary representations \(\hat{U}\) and \(\hat{V}\) of \(\hat{A}\) on finite dimensional Hilbert spaces, \(K_0(\hat{U}) = K_0(\hat{V})\) if, and only if, \(\hat{U}\) is unitarily equivalent to \(\hat{V}\). Furthermore, for any \(x \in K_0(A)^+\), there exists a unitary representation \(\hat{U}\) of \(\hat{A}\) with \(x = K_0(\hat{U})\), and for any two representations \(\hat{U}\) and \(\hat{V}\) of \(\hat{A}\), \(K_0(\hat{U}) \cdot K_0(\hat{V}) = K_0(\hat{U} \otimes \hat{V})\).
Actions and smash products with finite dimensional quantum groups

Definition
Let $A$ be a finite dimensional quantum group and let $R$ be an AF $C^*$-algebra. An action of $A$ on $R$ is a left $A$ module structure on $R$ such that for all $x, y \in R$ and $a \in A$, we have
$$a \cdot (xy) = m(\Delta(a) \cdot (x \otimes y)),$$
where $m$ denotes the multiplication map $m : R \otimes R \rightarrow R$ and $A \otimes A$ acts on $R \otimes R$ by the tensor product action. We shall call a triple $(R, A, \triangleright)$ where $R$ is a $C^*$-algebra, $A$ is a finite dimensional quantum group, and $\triangleright$ is an action of $A$ on $R$ a Hopf $C^*$-dynamical system. Given two Hopf $C^*$-dynamical systems $(R_1, A, \triangleright_1)$ and $(R_2, A, \triangleright_2)$ with the same quantum group, we define a morphism from $(R_1, A, \triangleright_1)$ to $(R_2, A, \triangleright_2)$ to be a $*$-homomorphism $\varphi : R_1 \rightarrow R_2$ such that $\varphi(a \cdot r) = a \cdot \varphi(r)$ for all $a \in A$ and $r \in R_1$. In agreement with the case of group actions, we shall also call such a map an equivariant $*$-homomorphism.
Definition
Given an action of a finite dimensional quantum group $A$ on an AF $C^*$-algebra $R$, we say that the action is representable (inner) if there exists a $*$-homomorphism $\gamma : A \to M(R)$, such that for all $a \in A$ and $x \in R$, we have $a \cdot x = \sum \gamma(a_{(1)})x\gamma(S(a_{(2)}))$, where $M(R)$ denotes the multiplier algebra of $R$ and we have used the Sweedler notation in the sum.
Definition
Given an action of a finite dimensional quantum group $A$ on a $C^*$-algebra $R$, a covariant representation of $(R, A)$ consists of a Hilbert space $H$, and *-homomorphisms $\pi : R \to B(H)$ and $\gamma : A \to B(H)$ such that $\pi(a \cdot x) = \sum \gamma(a(1))\pi(x)\gamma(S(a(2)))$ for all $x \in R$ and $a \in A$.

Definition
Given an action of a finite dimensional quantum group $A$ on an AF $C^*$-algebra $R$, we make $R \otimes A$ into an algebra by $(x \otimes a)(x' \otimes a') = (x \otimes 1)(\Delta(a) \cdot (x' \otimes a'))$, where $A \otimes A$ acts on $R \otimes A$ via the tensor product of its actions on itself and $R$, and extending linearly. With this multiplication, we denote $R \otimes A$ by $R\#A$. 
Lemma

If $R$ is a $C^*$-algebra and $A$ is a finite dimensional quantum group with an inner action on $R$ given by a $^*$-homomorphism $\gamma : A \to M(R)$, then $R \# A \cong R \otimes A$.

Proof.

In [DDZ] it is shown that then maps $\delta : R \# A \to R \otimes A$ given by $\delta(r \# a) = \sum r \gamma(a_{(1)}) \otimes a_{(2)}$ and $\lambda : R \otimes A \to R \# A$ given by $\lambda(r \otimes a) = \sum r \gamma(S(a_{(1)})) \# a_{(2)}$ are inverse algebra isomorphisms.
Lemma
Given two Hopf C*-dynamical systems $(R_1, A, ▶_1)$ and $(R_2, A, ▶_2)$ and an equivariant *-homomorphism $\varphi : R_1 \to R_2$, there exists a unique homomorphism, which we shall denote $\tilde{\varphi}$, from $R_1#A$ to $R_2#A$ such that $\tilde{\varphi}(r#a) = (\varphi(r))#a$ for all $r \in R_1$ and $a \in A$.

Lemma
Let $A$ be a finite dimensional quantum group acting representably on a finite dimensional C*-algebra $R$, let $V$ be a finitely generated projective left $A$ module, and let $W$ be a finitely generated projective left $R#A$ module. Then $W \otimes_\mathbb{C} V$ becomes a left $R#A$ module via $(r#a) \cdot (w \otimes v) = \sum ((r#a_{(1)}) \cdot w) \otimes (a_{(2)} \cdot v)$, and with this module structure, $W \otimes_\mathbb{C} V$ is finitely generated and projective. Furthermore, if $V' \cong V$ as a left $A$ module and $W' \cong W$ as a left $R#A$ module, then $W \otimes_\mathbb{C} V \cong W' \otimes_\mathbb{C} V'$ as a left $R#A$ module.
Definition
Let $A$ be a finite dimensional quantum group acting on a $C^*$-algebra $R$. We define an operation $K_0(R\#A) \times K_0(A) \rightarrow K_0(R\#A)$ as follows. Let $V$ be a finitely generated left $A$ module, and let $W$ be a finitely generated left $R\#A$ module. Set $[W] \cdot [V] = [W \otimes_C V]$, where the action of $R\#A$ on $W \otimes_C V$ is as in the lemma above. Now extend additively. The operation $\cdot$ satisfies $[W] \cdot ([M] [N]) = ([W] \cdot [M]) \cdot [N]$, and so defines a right $K_0(A)$ module structure on $K_0(R\#A)$. 
Lemma
Suppose that $R$ is a full matrix algebra and that $A$ acts on it representably. Then the isomorphism $R\#A \cong R \otimes A$ from above induces an isomorphism $K_0(R\#A) \cong K_0(R \otimes A) \cong K_0(A)$ of right $K_0(A)$ modules.

Lemma
Let $(R_1, A, ▶_1)$ and $(R_2, A, ▶_2)$ be two Hopf C* -dynamical systems with $R_1$, $R_2$, $A$ finite dimensional and the actions representable, and let $\varphi : R_1 \rightarrow R_2$ be an equivariant *-homomorphism. Then $K_0(\bar{\varphi}) : K_0(R_1\#A) \rightarrow K_0(R_2\#A)$ is a right $K_0(A)$ module map.
The following definition gives a $K_0$ that will play a role similar to that of the class of the unit for AF algebras.

**Definition**
Given a finite dimensional quantum group $A$ acting on a $C^*$-algebra $R$, we may define an operation $\cdot : (R \# A) \otimes R \to R$ by $(r_1 \# a) \otimes r_2 \mapsto r_1 (a \cdot r_2)$. This operation makes $R$ into a left $R \# A$ module that is finitely generated and projective. We write $\tilde{R}$ for $R$ with this module structure.
For the next theorem, we observe that $S$ respects equivalence of projections. If we extend $S$ to matrix algebras over $A$ by $x \otimes a \mapsto x^T \otimes S(a)$, where $T$ denotes the transpose of a matrix, we see that this extension also respects equivalence of projections. We can thus define a period two automorphism of $K_0(A)$ by $S_*([p]) = [S(p)]$ for $p \in M_n(A)$.

**Theorem**

*Let $A$ be a finite dimensional quantum group acting on a full matrix algebra $R$, with action coming from a unitary representation $\hat{U}$ of $\hat{A}$. Then, under the identification of $K_0(R\#A)$ with $K_0(A)$ coming from the isomorphism of $R\#A$ with $R \otimes A$ from above, $[\tilde{R}] = nS_*K_0(\hat{U})$, where $n$ is the size of the matrix algebra $R$.***
Definition
Given two actions $\triangleright_1$ and $\triangleright_2$ of a finite dimensional quantum group $A$ on unital AF $C^*$-algebras $R_1$ and $R_2$ respectively, the tensor product action $\triangleright_1 \otimes \triangleright_2$ of $A$ on $R_1 \otimes R_2$ is defined by
\[ a \cdot (x \otimes y) = \sum (a_{(1)} \triangleright_1 x) \otimes (a_{(2)} \triangleright_2 y). \] (cf. [DDZ])

Theorem
Let $\triangleright_1$ and $\triangleright_2$ be two actions of a finite dimensional quantum group $A$ on full matrix algebras $R_1$ and $R_2$ respectively, corresponding to unitary representations $\hat{U}_1$ and $\hat{U}_2$ of $\hat{A}$. Then $\triangleright_1 \otimes \triangleright_2$ is conjugate to the inner action defined by $\hat{U}_1 \otimes \hat{U}_2$. 
Existence and uniqueness theorems

Theorem

(Existence) Let \((R_1, A, \triangleright_1)\) and \((R_2, A, \triangleright_2)\) be two representable Hopf \(C^*\)-dynamical systems, with \(R_1\) and \(R_2\) finite dimensional, and with the same finite dimensional quantum group \(A\) acting, and suppose that \(\psi\) is an ordered \(K_0(A)\) module homomorphism from \(K_0(R_1\#A)\) to \(K_0(R_2\#A)\) such that \(\psi([\tilde{R}_1]) = [\tilde{R}_2]\). Then there exists a unital equivariant \(*\)-homomorphism \(\tilde{\psi} : R_1 \to R_2\) such that \(\psi = K_0(\tilde{\psi})\).

Theorem

(Uniqueness) Let \((R_1, A, \triangleright_1)\) and \((R_2, A, \triangleright_2)\) be two representable Hopf \(C^*\)-dynamical systems, with \(R_1\) and \(R_2\) finite dimensional, and with the same finite dimensional quantum group \(A\) acting, and let \(\psi\) and \(\phi\) be two equivariant homomorphisms from \(R_1\) to \(R_2\). If \(K_0(\tilde{\phi}) = K_0(\tilde{\psi})\), as maps from \(K_0(R_1\#A)\) to \(K_0(R_2\#A)\), then there exists an equivariant inner automorphism \(\alpha : R_2 \to R_2\) such that \(\psi = \alpha \circ \phi\).
Definition
Let \( A \) be a finite dimensional quantum group and let \( R \) be a unital AF \( C^* \)-algebra. An action of \( A \) on \( R \) will be called locally representable if there exists a sequence \( \{ R_n \} \) of finite dimensional sub-\( C^* \)-algebras of \( R \) such that \( \bigcup_n R_n = R \), \( 1 \in R_n \subseteq R_{n+1} \) for all \( n \), each \( R_n \) is globally invariant under the action of \( A \) on \( R \), and the restriction of the action of \( A \) to \( R_n \) is representable (inner).
In order to work with inductive limits, it is convenient to make our
smash products into $C^*$-algebras, which the following theorem lets
us do.

**Theorem**

Let $A$ be a finite dimensional quantum group acting locally
representably on a unital AF $C^*$-algebra $R$, and let \( \{R_n\} \) be a
sequence of finite dimensional sub-$C^*$-algebras of $R$ satisfying the
conditions in the definition of local representability. Define an
operation \( x \mapsto x^\dagger \) on $R \# A$ by \( (r \# a)^\dagger = \sum a_{(1)}^* \cdot r^* \# a_{(2)}^* \) and
extending additively. Then, with $\dagger$ as adjoint operation, $R \# A$ is an
AF $C^*$-algebra, each $R_n \# A$ is self adjoint, and $R \# A = \bigcup_n (R_n \# A)$. 
The proof of our main theorem follows the familiar pattern of Elliott intertwining arguments.

**Theorem**

Let \( \{(R_n, A, \triangleright_n), \varphi_{nm}\} \) and \( \{(R'_n, A, \triangleright'_n), \varphi'_{nm}\} \) be two directed systems of Hopf C\(^*\)-dynamical systems where \( A \) is a finite dimensional quantum group, each \( R_n \) and \( R'_n \) is finite dimensional, and each action is inner. Let \((R, A, \triangleright)\) and \((R', A, \triangleright')\) denote the inductive limit Hopf C\(^*\)-dynamical systems respectively. Suppose that \( \psi \) is an ordered \( K_0(A) \) module isomorphism of \( K_0(R\#A) \) with \( K_0(R'\#A) \) that takes \([\tilde{R}]\) to \([\tilde{R}']\). Then there exists an equivariant isomorphism \( \tilde{\psi} \) of \((R, A, \triangleright)\) with \((R', A, \triangleright')\) such that \( \psi = K_0(\tilde{\psi}) \).
Example: The Kac-Paljutkin algebra

There exists a unique non-commutative, non-co-commutative, eight dimensional quantum group, discovered by Kac and Paljutkin, which we shall denote $KP$. See [EV] for a discussion of this quantum group. As a $C^*$ algebra, it is isomorphic to $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$. Writing $e_1, e_2, e_3, e_4, e_{11}, e_{22}, e_{12}, e_{21}$ for a system of matrix units, the co-product, $\Delta$, is given by:
\[ \Delta(e_1) = e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 + e_4 \otimes e_4 + \frac{1}{2}(e_{11} \otimes e_{11} + e_{22} \otimes e_{22} + e_{12} \otimes e_{12} + e_{21} \otimes e_{21}), \]

\[ \Delta(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 + e_3 \otimes e_4 + e_4 \otimes e_3 + \frac{1}{2}(e_{11} \otimes e_{22} + e_{22} \otimes e_{11} + ie_{12} \otimes e_{21} - ie_{21} \otimes e_{12}), \]

\[ \Delta(e_3) = e_1 \otimes e_3 + e_2 \otimes e_4 + e_3 \otimes e_1 + e_4 \otimes e_2 + \frac{1}{2}(e_{11} \otimes e_{11} + e_{22} \otimes e_{22} - e_{12} \otimes e_{12} - e_{21} \otimes e_{21}), \]

\[ \Delta(e_4) = e_1 \otimes e_4 + e_2 \otimes e_3 + e_3 \otimes e_2 + e_4 \otimes e_1 + \frac{1}{2}(e_{11} \otimes e_{22} + e_{22} \otimes e_{11} - ie_{12} \otimes e_{21} + ie_{21} \otimes e_{12}), \]
\[ \Delta(e_{11}) = e_1 \otimes e_{11} + e_2 \otimes e_{22} + e_3 \otimes e_{11} + e_4 \otimes e_{22} + e_{11} \otimes e_1 + e_{22} \otimes e_2 \\
+ e_{11} \otimes e_3 + e_{22} \otimes e_4, \]

\[ \Delta(e_{12}) = e_1 \otimes e_{12} - ie_2 \otimes e_{21} - e_3 \otimes e_{12} + ie_4 \otimes e_{21} + e_{12} \otimes e_1 + ie_{21} \otimes e_2 \\
- e_{12} \otimes e_3 - ie_{21} \otimes e_4, \]

\[ \Delta(e_{21}) = e_1 \otimes e_{21} + ie_2 \otimes e_{12} - e_3 \otimes e_{21} - ie_4 \otimes e_{12} + e_{21} \otimes e_1 - ie_{12} \otimes e_2 \\
- e_{21} \otimes e_3 + ie_{12} \otimes e_4, \]

and

\[ \Delta(e_{22}) = e_1 \otimes e_{22} + e_2 \otimes e_{11} + e_3 \otimes e_{22} + e_4 \otimes e_{11} + e_{22} \otimes e_1 + e_{11} \otimes e_2 \\
+ e_{22} \otimes e_3 + e_{11} \otimes e_4. \]
The antipode, \( S \), is given by: \( S(e_i) = e_i \) for \( i = 1, \ldots, 4 \) and \( S(e_{ij}) = e_{ji} \) for \( i, j = 1, 2 \).

Since the co-product is non-symmetric, the dual quantum group is non-commutative, and it is in fact isomorphic to \( KP \). The ring structure on \( K_0(KP) \) is given by the following multiplication table, where 1 denotes \([e_1]\):

<table>
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<tr>
<th></th>
<th>1</th>
<th>[e_2]</th>
<th>[e_3]</th>
<th>[e_4]</th>
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<tr>
<td>1</td>
<td>1</td>
<td>[e_2]</td>
<td>[e_3]</td>
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<tr>
<td>[e_2]</td>
<td>[e_2]</td>
<td>1</td>
<td>[e_4]</td>
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<td>[e_4]</td>
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<td>[e_{11}]</td>
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<td>[e_{11}]</td>
<td>[e_{11}]</td>
<td>1 + [e_2] + [e_3] + [e_4]</td>
</tr>
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</table>

Notice that \( K_0(KP) \) is a commutative ring.
We consider the action of $KP$ on the UHF algebra $M_{2\infty}$ arising from repeatedly tensoring with the representation corresponding to $\pi_{2\times 2}$. More precisely, $R_n = M_{2^n}$, $\varphi_n = \varphi U \otimes \cdots \otimes U$, where $U$ is the representation corresponding to $\pi_{2\times 2}$, and the connecting maps are $x \mapsto x \otimes 1$. From the multiplication table above, we get that the repeated matrix for our inductive system is the matrix $A$ shown below. Considering $A^2$ makes it easy to see that $K_0(M_{2\infty} \# KP) \cong \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}[\frac{1}{2}]$. 
\[ A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \quad A^2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \]
The class of the special element is $(0, 1)$ with the identification above. It remains to determine the $K_0(KP)$ module structure. Our inductive system is

$$K_0(KP) \xrightarrow{[e_{11}] \times} K_0(KP) \xrightarrow{[e_{11}] \times} K_0(KP) \xrightarrow{[e_{11}] \times} K_0(KP) \rightarrow \ldots$$

Since multiplication by $[e_{11}]$ absorbs multiplication by each $[e_i]$, we see that the elements $[e_i]$ each act as the identity. Multiplication by $[e_{11}]$ gives the map $(a, b) \mapsto (4b, a)$. 
References

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