SPECTRAL RADIUS ALGEBRAS OF WEIGHTED CONDITIONAL EXPECTATION OPERATORS ON $L^2(\mathcal{F})$

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The Operators under investigation act on the Hilbert space $L^2(\mathcal{F})$ and are of the form M_wEM_u, where M_w and M_u are (not necessarily bounded) *multiplication operators* with symbols u and w, and E is the *conditional expectation operator* relative to σ-subalgebra $\mathcal{A}$.

From now on we use the term "*conditional type* operators" in stead of "*weighted conditional expectation* operators".

The *conditional type* operators in function spaces were studied by many authors. Here we recall some of them.
In the first investigation Moy obtained necessary and sufficient conditions for a linear transformation T between function spaces to be of the form of *conditional type* operators.

- Shu-Teh Chen, Moy, Characterizations of conditional expectation as a transformation on function spaces, Pacific J. Math. **4** (1954), 47-63.

Grobler and de Pagter showed that class of *partial integral operators* are *conditional type* operators.

- J. J. Grobler and B. de Pagter, Operators representable as multiplication-conditional expectation operators, J. Operator Theory **48** (2002), 15-40.
- P.G. Dodds, C.B. Huijsmans and B. De Pagter, characterizations of conditional expectation-type operators, Pacific J. Math. **141** (1990), 55-77.
- J. Herron, Weighted conditional expectation operators, Oper. Matrices **1** (2011), 107-118.

Also, we investigated some classical properties of *conditional type* operators on L^p-spaces.

- Y. Estaremi and M.R. Jabbarzadeh, Weighted lambert type operators on L^p-spaces, Oper. Matrices **1** (2013), 101-116.
- Y. Estaremi, Some classes of weighted conditional type operators and their spectra, Positivity. **19** (2015) 83-93.
- Y. Estaremi, Unbounded weighted conditional expectation operators, Complex Anal. Oper. Theory **10** (2016) 567580.

For a σ- subalgebra $\mathcal{A}$ of $\mathcal{F}$, the conditional expectation operator associated with $\mathcal{A}$ is the mapping $f \to E^{\mathcal{A}} f$, defined for all non-negative measurable function f as well as for all $f \in L^2(\mathcal{F})$, where $E^{\mathcal{A}} f$, by the Radon-Nikodym theorem, is the unique $\mathcal{A}$-measurable function satisfying

$$\int_A f d\mu = \int_A E^{\mathcal{A}} f d\mu, \quad \forall A \in \mathcal{A}.$$

As an operator on $L^2(\mathcal{F})$, $E^{\mathcal{A}}$ is idempotent and $E^{\mathcal{A}}(L^2(\mathcal{F})) = L^2(\mathcal{A})$. We will often write E for $E^{\mathcal{A}}$.

This operator will play a major role in our work and we list here some of its useful properties:

- If g is $\mathcal{A}$-measurable, then $E(fg) = E(f)g$.
- If $f \geq 0$, then $E(f) \geq 0$; if $E(|f|) = 0$, then $f = 0$.
- $|E(fg)| \leq (E(|f|^2))^{\frac{1}{2}}(E(|g|^2))^{\frac{1}{2}}$.
- For each $f \geq 0$, $z(E(f))$ is the smallest $\mathcal{A}$-set containing $z(f)$, where $z(f) = \{x \in X : f(x) \neq 0\}$.

A detailed discussion and verification of most of these properties may be found in

- M. M. Rao, Conditional measure and applications, Marcel Dekker, New York, 1993.

Here we will briefly review some basic facts about spectral radius algebras. Let $\mathcal{H}$ be a Hilbert space, $T \in \mathcal{B}(\mathcal{H})$ and $r(T)$ be the spectral radius of T. For $m \geq 1$ we define

$$R_m(T) = R_m := \left(\sum_{n=0}^{\infty} d_m^{2n} T^{*^n} T^n \right)^{\frac{1}{2}}, \qquad (1)$$

where $d_m = \frac{1}{1/m + r(T)}$. Since $d_m \uparrow 1/r(T)$, the sum in (1) is norm convergent and the operators R_m are well defined, positive and invertible.

Let $\mathcal{B}_T$ be the set of all operators $S \in \mathcal{B}(\mathcal{H})$ such that

$$\sup_{m \in \mathbb{N}} \|R_m S R_m^{-1}\| < \infty.$$

Clearly $\mathcal{B}_T$ is an algebra and contains all operators commute with T.

Interested people can find more details in

- A. Lambert and S. Petrovic, Beyond hyperinvariance for compact operators, J. Functional Analysis, (2005).
- A. Biswas, A. Lambert and S. Petrovic, On spectral radius algebras and normal operators, In. Univ. Math. J. (2007).
- A. Biswas, A. Lambert, S. Petrovic and B. Weinstock, On spectral radius algebras, Oper. Matrices. (2008).

It appears to be quite difficult to find explicit descriptions of of the operators in $\mathcal{B}_T$ for a given operator. We now illustrate the level of difficulty one should expect by describing a spectral algebra for a particularly simple operator.

Let $T = M_w E M_u$ be a bounded operator on $L^2(\mathcal{F})$. Direct computations shows that for every $n \in \mathbb{N}$(natural numbers) we have

$$T^n f = (E(uw))^{n-1} w E(uf), \qquad T^{*^n} f = (\overline{E(uw)})^{n-1} \bar{u} E(\bar{w} f).$$

Hence we get the positive, invertible operator R_m for $M_w E M_u$ as follows:

$$R_m = \left(I + M_{(E(|w|^2) \sum_{n=1}^{\infty} d_m^{2n} |E(uw)|^{2(n-1)})} M_{\bar{u}} E M_u \right)^{\frac{1}{2}} .$$

It is easy to see that the following equality holds almost every where on X.

$$\sum_{n=1}^{\infty} d_m^{2n}|E(uw)|^{2(n-1)} = \frac{d_m^2}{1-d_m^2|E(uw)|^2}.$$

If we set

$$v_m = \frac{d_m^2 E(|w|^2)}{1-d_m^2|E(uw)|^2},$$

then we have

$$R_m = (I + M_{v_m\bar{u}} E M_u)^{\frac{1}{2}}.$$

By an elementary technical method we can compute the inverse of R_m as follow:

$$R_m^{-1} = \left(I + M_{\frac{v_m\bar{u}}{v_m E(|u|^2)-1}} E M_u\right)^{\frac{1}{2}}.$$

Here we recall a fundamental lemma in operator theory.

Lemma

Let T be a bounded operator on the Hilbert space $\mathcal{H}$ and $\lambda \geq 0$. Then we have

$$\|\lambda I + T^* T\| = \lambda + \|T^* T\| = \lambda + \|T\|^2.$$

Specially, if T is a positive operator, then $\|\lambda I + T\| = \lambda + \|T\|$.

From now on we assume that $E(|u|^2) \in L^\infty(\mathcal{A})$. Now we characterize the spectral radius algebra corresponding to the *conditional type* operator $M_w E M_u$ in the next theorem.

Theorem

Let $S \in \mathcal{B}(L^2(\mathcal{F}))$. Then $S \in \mathcal{B}_{M_w E M_u}$ if and only if $\mathcal{N}(EM_u)$ is invariant under S.

Therefore we get that there are many different operators that have the same spectral radius algebra.

Corollary

If $M_w EM_u$ and $M_{w'} EM_u$ are bounded operator on the Hilbert space $L^2(\mathcal{F})$, then $\mathcal{B}_{M_{w'} EM_u} = \mathcal{B}_{M_w EM_u}$.

Also we have a sufficient condition for $\mathcal{B}_{M_w EM_u}$ to be equal to $\mathcal{B}(L^2(\mathcal{F}))$.

Corollary

If $\mathcal{N}(EM_u) = \{0\}$, then $\mathcal{B}_{M_w EM_u} = \mathcal{B}(L^2(\mathcal{F}))$.

Proposition

If $a \in L^0(\mathcal{A})$ such that $a \geq 0$ and $M_{a\bar{u}}EM_u \in \mathcal{B}(L^2(\mathcal{F}))$, then $M_{a\bar{u}}EM_u \in \mathcal{B}_{M_w EM_u}$.

Every operator T on a Hilbert space $\mathcal{H}$ can be decomposed into $T = U|T|$ with a partial isometry U, where $|T| = (T^*T)^{\frac{1}{2}}$. U is determined uniquely by the kernel condition $\mathcal{N}(U) = \mathcal{N}(|T|)$. Then this decomposition is called the polar decomposition. The Aluthge transformation $\widehat{T}$ of the operator T is defined by $\widehat{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$.

Here we recall that the Aluthge transformation of $T = M_w E M_u$ is

$$\widehat{T}(f) = \frac{\chi_{z_1} E(uw)}{E(|u|^2)} \bar{u} E(uf), \qquad f \in L^2(\mathcal{F}).$$

in which $z_1 = z(E(|u|^2))$. Thus $\widehat{T} = M_{w'} E M_{u'}$ where $w' = \frac{E(uw)\bar{u}\chi_{z_1}}{E(|u|^2)}$ and $u' = u$. We recall that $r(M_w E M_u) = \|E(uw)\|_\infty$. Direct computations shows that $E(u'w') = E(uw)$. Hence $r(T) = r(\widehat{T})$. Hence we have the next corollary.

Corollary

If w and u are positive measurable functions, then $\widehat{T} \in \mathcal{B}_T$ where $T = M_w E M_u$.

Moreover we get that the commutant of $M_w EM_u$ (in symbol $\{M_w EM_u\}'$) is a proper subset of $\mathcal{B}_{M_w EM_u}$ when w, u are positive and $w \neq u$. In the next corollary we get that $\mathcal{B}_T = \mathcal{B}_{\widehat{T}}$ when $T = M_w EM_u$ and $w, u \geq 0$.

Corollary

If $T = M_w EM_u$ and $w, u \geq 0$, then $\mathcal{B}_T = \mathcal{B}_{\widehat{T}}$.

Remark

Suppose that $T = EM_u \in \mathcal{B}(L^2(\mathcal{F}))$, $u \in L^\infty(\mathcal{A})$ and $\mathcal{A}, \mathcal{B}$ be σ-subalgebras of $\mathcal{F}$ such that $\mathcal{A} \subseteq \mathcal{B}$. If $E = E^{\mathcal{A}}$ and S is an operator for which $TS = E^{\mathcal{B}}ST$, then $S \in \mathcal{B}_T$.

Here we recall the definition of Q_T for $T \in \mathcal{B}(\mathcal{H})$, that is defined in

- A. Lambert and S. Petrovic, Beyond hyperinvariance for compact operators, J. Functional Analysis, (2005),

as follows:

$$Q_T = \{S \in \mathcal{B}(\mathcal{H}) : \|R_m S R_m^{-1}\| \to 0\}.$$

Q_T is a two sided ideal in $\mathcal{B}_T$ and every operator in Q_T is quasi-nilpotent. In the next Theorem we illustrate Q_T when $T = M_w E M_u \in \mathcal{B}(L^2(\mathcal{F}))$.

Theorem

Let $T = M_w E M_u$ and $S \in \mathcal{B}(L^2(\mathcal{F}))$. Then $S \in Q_T$ if and only if $\mathcal{N}(EM_u)$ is invariant under S and $\mathcal{N}(EM_u) \subseteq \mathcal{N}(S)$.

Now we have an equivalent condition for the spectral radius algebra of a *conditional type* operator to be equal to $\mathcal{B}(L^2(\mathcal{F}))$.

Proposition

If $T = M_w E M_u$, then $\mathcal{B}_T = \mathcal{B}(L^2(\mathcal{F}))$ if and only if

$$\sup_m(\|E(|u|^2)v_m\|_\infty + \|\frac{E(|u|^2)v_m}{v_m E(|u|^2) - 1}\|_\infty(1 + \|E(|u|^2)v_m\|_\infty)) < \infty$$

where $v_m = \frac{d_m^2 E(|w|^2)}{1 - d_m^2 |E(uw)|^2}$.

Also, we have an equivalent condition for the *conditional type* operator $M_w E M_u$ to be a constant multiple of an isometry.

Theorem

If $T = M_w E M_u$ is a bounded operator on the Hilbert space $L^2(\mathcal{F})$, then T is a constant multiple of an isometry if and only if

$$\sup_m (\|E(|u|^2)v_m\|_\infty + \|\frac{E(|u|^2)v_m}{v_m E(|u|^2) - 1}\|_\infty (1 + \|E(|u|^2)v_m\|_\infty)) < \infty$$

where $v_m = \frac{d_m^2 E(|w|^2)}{1 - d_m^2 |E(uw)|^2}$.

Recall that for $f, g \in L^2(\mathcal{F})$ we can define a rank one operator $f \otimes g$ on $L^2(\mathcal{F})$ by the action $(f \otimes g)(h) = \langle h, g \rangle f$ for every $h \in L^2(\mathcal{F})$, in which $\langle , \rangle$ is the inner product of the Hilbert space $L^2(\mathcal{F})$. In the next proposition we give some conditions under which a rank one operator belongs to the $\mathcal{B}_{M_w E M_u}$.

Proposition

If $T = M_w E M_u$ and $f, g \in L^2(\mathcal{F})$, then $f \otimes g \in \mathcal{B}_T$ if and only if

$$\sup_m \|\alpha_m^{\frac{1}{2}} E(ug)\|^2 \|f\|^2 + \|v_m^{\frac{1}{2}} E(uf)\|^2 (\|g\|^2 + \|\alpha_m^{\frac{1}{2}} E(ug)\|^2) < \infty,$$

where $\alpha_m = \frac{v_m}{v_m E(|u|^2) - 1}$.

Proposition, A. Lambert and S. Petrovic, 2005

If $x, y \in \mathcal{H}$ are unit vectors, then $S \in \mathcal{B}_{x \otimes y}$ if and only if y is an eigenvector for S^*.

Remark

For the unit vectors u, v, w of the Hilbert space $\mathcal{H}$ we have $\mathcal{B}_{u\otimes w} = \mathcal{B}_{v\otimes w}$.

In the next Theorem we describe $Q_{u\otimes v}$ for a rank one operator $u \otimes v$ in which u, v are in the Hilbert space $\mathcal{H}$.

Theorem

Let $\mathcal{H}$ be a Hilbert space and $S \in \mathcal{B}(\mathcal{H})$. If $u, v \in \mathcal{H}$, then $S \in Q_{u\otimes v}$ if and only if $S = (I - P)SP$, where $P = P_{\mathcal{H}_1}$ and $\mathcal{H}_1$ is the one-dimensional space spanned by v.

Let X, Y, Z be Banach spaces. Assume that $T \in \mathcal{B}(X, Y)$ and $S \in \mathcal{B}(X, Z)$. Then T majorizes S if there exists $M > 0$ such that

$$\|Sx\| \leq M\|Tx\|$$

for all $x \in X$. Here we recall a Proposition that gives us an equivalent condition for a closed range operator to majorize another bounded operator.

Remark

Let X be Banach spaces and $T, S \in \mathcal{B}(X)$ with $\mathcal{R}(T)$ closed. Then T majorizes S if and only if $\mathcal{N}(T) \subseteq \mathcal{N}(S)$.

Proposition

Let $T = M_w E M_u$ and $u \geq 0$. If $S \in Q_T$ and $E(u) \geq \delta$ a.e., then EM_u majorizes S.

Finally, since the rank one operator $x \otimes y$ has closed range, the we can obtain the next proposition.

Proposition

Let $x, y \in \mathcal{H}$. If $T \in Q_{x \otimes y}$, then $x \otimes y$ majorizes T.

Thank You For Your Attention!