Operator Space Projective Tensor Product: Embedding into second dual and ideal structures

Ranjana Jain
(Joint Work with Ajay Kumar)

University of Delhi, INDIA

August 15, 2016
Conference on “Groups and Operators”
Chalmers University of Technology and University of Gothenburg
SWEDEN
Overview

- Operator Spaces
- Tensor Products
- Operator space projective tensor product
- Embeddings of tensor products into bidual
- Algebraic structure
An (concrete) operator space \( V \) is a subspace of \( B(H) \) together with the natural norms on \( M_n(V) \) inherited from \( M_n(B(H)) = B(H^n) \), \( H \) being a Hilbert space.

Every Banach space possesses an operator space structure. To see this, for \( \Gamma = B_1(X^*) \), the isometric embedding \( X \hookrightarrow C(\Gamma) \) via \( x \rightarrow f_x \), where \( f_x(g) := g(x) \) for \( g \in C(\Gamma) \), equips \( X \) with an operator space structure.

Every \( C^* \)-algebra is an operator space.

For Hilbert spaces \( H \) and \( K \), \( B(H, K) \subseteq B(H \oplus K) \) is an operator space.
A linear map \( \phi : V \rightarrow W \) between operator spaces \( V \) and \( W \) is said to be completely bounded (in short, c.b.) if
\[
\| \phi \|_{cb} := \sup \{ \| \phi_n \| : n \in \mathbb{N} \} < \infty,
\]
where \( \phi_n : M_n(V) \rightarrow M_n(W) \) is defined by
\[
\phi_n((x_{ij})) = (\phi(x_{ij})) \text{ for all } (x_{ij}) \in M_n(V).
\]

Two operator spaces \( V \) and \( W \) are said to be completely isomorphic if there exists a completely bounded linear bijection \( \phi : V \rightarrow W \) whose inverse is also completely bounded.
Definition:
A normed space $V$ with a sequence of norms $\| \cdot \|_n : M_n(V) \to [0, \infty)$, $n \in \mathbb{N}$
is said to be an (abstract) operator space if:

(i) $\| \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix} \|_{n+m} = \max\{\|v\|_n, \|w\|_m\}$ for all $v \in M_n(V), w \in M_m(V)$

(ii) $\| \alpha v \beta \|_m \leq \|\alpha\| \|v\|_n \|\beta\|$, for all $\alpha \in M_{m,n}, \beta \in M_{n,m}, v \in M_n(V)$.

Ruan, 1988

If $V$ is an abstract operator space, then $V$ is completely isometrically isomorphic to a linear subspace of $B(H)$ for some Hilbert space $H$.

Definition:

A normed space $V$ with a sequence of norms

$$
\| \cdot \|_n : M_n(V) \to [0, \infty), \quad n \in \mathbb{N}
$$

is said to be an (abstract) operator space if:

(i) $\|(v \ 0 \ 0 \ w)\|_{n+m} = \max\{\|v\|_n, \|w\|_m\}$ for all $v \in M_n(V), w \in M_m(V)$

(ii) $\|\alpha v \beta\|_m \leq \|\alpha\| \|v\|_n \|\beta\|$, for all $\alpha \in M_{m,n}, \beta \in M_{n,m}, v \in M_n(V)$.

Ruan, 1988

If $V$ is an abstract operator space, then $V$ is completely isometrically isomorphic to a linear subspace of $B(H)$ for some Hilbert space $H$.

(c.b. maps) $\mathcal{CB}(V, W)$, the space of completely bounded maps from $V$ into $W$ is an operator space with the matricial norms inherited from the identifications

$$M_n(\mathcal{CB}(V, W)) \cong \mathcal{CB}(V, M_n(W))$$

(Dual) The operator space structure on the dual $V^*$ comes by identifying $V^* = \mathcal{CB}(V, \mathbb{C}) \cong \mathcal{B}(V, \mathbb{C})$. Thus, $V^*$ obtains its matricial structure from the identifications

$$M_n(V^*) \cong \mathcal{CB}(V, M_n(\mathbb{C})),$$

and the operator space thus obtained is known as the operator space dual of $V$. 
Tensor Norms/Products

- For $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$, Banach space injective tensor norm is defined as

$$\|u\|_{\lambda} = \sup \left\{ \left| \sum_{i=1}^{n} f(x_i)g(y_i) \right| : f \in B_1(X^*), g \in B_1(Y^*) \right\}.$$ 

- Let $X$ and $Y$ be Banach spaces. For $u \in X \otimes Y$, Banach space projective tensor norm is defined as

$$\|u\|_{\gamma} = \inf \left\{ \sum_{i=1}^{n} \|x_i\|\|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i \right\},$$

For a bilinear map $u : V \times W \to E$, there are two ways to look at its amplifications

$$u_n : M_n(V) \times M_n(W) \to M_n(E)$$

$$u_n((v_{ij}), (w_{kl})) = \left( \sum_{k=1}^{n} u(v_{ik}, w_{kj}) \right), \ n \in \mathbb{N}$$

$$u'_n : M_n(V) \times M_n(W) \to M_{n^2}(E)$$

$$u'_n((v_{ij}), (w_{kl})) = (u(v_{ij}, w_{kl})), \ n \in \mathbb{N}$$

- $u$ is completely bilinear (c.b.) if $u_n$ are uniformly bounded, and we denote $\|u\|_{cb} = \sup_n \|u_n\|$.

- $u$ is jointly completely bilinear (j.c.b.) if $u'_n$ are uniformly bounded, and we denote $\|u\|_{jcb} = \sup_n \|u'_n\|$. 
For a bilinear map $u : V \times W \to E$, there are two ways to look at its amplifications

$u_n : M_n(V) \times M_n(W) \to M_n(E)$

$$u_n((v_{ij}), (w_{kl})) = \left( \sum_{k=1}^{n} u(v_{ik}, w_{kj}) \right), \ n \in \mathbb{N}$$

$u'_n : M_n(V) \times M_n(W) \to M_{n^2}(E)$

$$u'_n((v_{ij}), (w_{kl})) = (u(v_{ij}, w_{kl})), \ n \in \mathbb{N}$$

- $u$ is completely bilinear (c.b.) if $u_n$ are uniformly bounded, and we denote $\|u\|_{cb} = \sup_n \|u_n\|$.
- $u$ is jointly completely bilinear (j.c.b.) if $u'_n$ are uniformly bounded, and we denote $\|u\|_{jcb} = \sup_n \|u_n\|$.
For a bilinear map $u : V \times W \rightarrow E$, there are two ways to look at its amplifications

$u_n : M_n(V) \times M_n(W) \rightarrow M_n(E)$

$$u_n((v_{ij}), (w_{kl})) = \left( \sum_{k=1}^{n} u(v_{ik}, w_{kj}) \right), \; n \in \mathbb{N}$$

$u'_n : M_n(V) \times M_n(W) \rightarrow M_{n^2}(E)$

$$u'_n((v_{ij}), (w_{kl})) = (u(v_{ij}, w_{kl})), \; n \in \mathbb{N}$$

- $u$ is completely bilinear (c.b.) if $u_n$ are uniformly bounded, and we denote $\|u\|_{cb} = \sup_n \|u_n\|$.

- $u$ is jointly completely bilinear (j.c.b.) if $u'_n$ are uniformly bounded, and we denote $\|u\|_{jcb} = \sup_n \|u_n\|$.
For a bilinear map $u : V \times W \to E$, there are two ways to look at its amplifications

$u_n : M_n(V) \times M_n(W) \to M_n(E)$

$$u_n((v_{ij}), (w_{kl})) = \left( \sum_{k=1}^{n} u(v_{ik}, w_{kj}) \right), \ n \in \mathbb{N}$$

$u'_n : M_n(V) \times M_n(W) \to M_{n^2}(E)$

$$u'_n((v_{ij}), (w_{kl})) = (u(v_{ij}, w_{kl})), \ n \in \mathbb{N}$$

$u$ is completely bilinear (c.b.) if $u_n$ are uniformly bounded, and we denote $\|u\|_{cb} = \sup_n \|u_n\|$. 

$u$ is jointly completely bilinear (j.c.b.) if $u'_n$ are uniformly bounded, and we denote $\|u\|_{jcb} = \sup_n \|u_n\|$. 

Ranjana Jain, India
Operator Space Projective Tensor Product
For operator spaces $V$ and $W$, the **Haagerup tensor norm** is defined as

$$
\|u\|_h = \inf \{ \|v\| \|w\| : u = v \circlearrowleft w, v \in M_{n,p}(V), w \in M_{p,n}(W), p \in \mathbb{N} \},
$$

where $u \in M_n(V \otimes W)$ and $v \circlearrowleft w = \left( \sum_{k=1}^p v_{ik} \otimes w_{kj} \right)_{ij}$.

- For $C^*$-algebras $A$ and $B$, the Haagerup norm of an element $u \in A \otimes B$ takes a simpler and convenient form given by

$$
\|u\|_h = \inf \left\{ \left\| \sum_i a_i a_i^* \right\|^{1/2} \left\| \sum_i b_i^* b_i \right\|^{1/2} : u = \sum_{i=1}^n a_i \otimes b_i \right\}.
$$

- $(V \otimes^h W)^* \cong CB(V \times W, \mathbb{C})$.

Effros and Kishimoto
Paulsen and Smith, 1987
Haagerup Tensor Product

**Definition**

For operator spaces $V$ and $W$, the *Haagerup tensor norm* is defined as

$$
\| u \|_h = \inf \{ \| v \| \| w \| : u = v \odot w, v \in M_{n,p}(V), w \in M_{p,n}(W), p \in \mathbb{N} \},
$$

where $u \in M_n(V \otimes W)$ and $v \odot w = \left( \sum_{k=1}^p v_{ik} \otimes w_{kj} \right)_{ij}$.

- For $C^*$-algebras $A$ and $B$, the Haagerup norm of an element $u \in A \otimes B$ takes a simpler and convenient form given by

  $$
  \| u \|_h = \inf \{ \| \sum_i a_i a_i^* \|^{1/2} \| \sum_i b_i^* b_i \|^{1/2} : u = \sum_{i=1}^n a_i \otimes b_i \}.
  $$

- $(V \otimes^h W)^* \cong \text{CB}(V \times W, \mathbb{C})$.

*Effros and Kishimoto
Paulsen and Smith, 1987*
Definition

The operator space projective tensor product denoted by $V \hat{\otimes} W$, is the completion of the algebraic tensor product $V \otimes W$ under the norm

$$\|u\|_\wedge = \inf \{ \|\alpha\|\|v\|\|w\|\|\beta\| : u = \alpha(v \otimes w)\beta \}, \ u \in M_n(V \otimes W),$$

where infimum runs over arbitrary decompositions with $v \in M_p(V)$, $w \in M_q(W)$, $\alpha \in M_{n,pq}$, $\beta \in M_{pq,n}$ and $p, q \in \mathbb{N}$ arbitrary.

$$(V \hat{\otimes} W)^* \overset{cb}{=} JCB(V \times W, \mathbb{C}) \overset{cb}{=} CB(V, W^*) \overset{cb}{=} CB(W, V^*)$$

Definition

The operator space projective tensor product denoted by $V \hat{\otimes} W$, is the completion of the algebraic tensor product $V \otimes W$ under the norm

$$\|u\|_\wedge = \inf \{ \|\alpha\| \|v\| \|w\| \|\beta\| : u = \alpha(v \otimes w)\beta \}, \ u \in M_n(V \otimes W),$$

where infimum runs over arbitrary decompositions with $v \in M_p(V)$, $w \in M_q(W)$, $\alpha \in M_{n,pq}$, $\beta \in M_{pq,n}$ and $p, q \in \mathbb{N}$ arbitrary.

$$(V \hat{\otimes} W)^* \overset{cb}{=} \text{JCB}(V \times W, \mathbb{C}) \overset{cb}{=} \text{CB}(V, W^*) \overset{cb}{=} \text{CB}(W, V^*)$$

On the tensor product $A \otimes B$ of two $C^*$-algebras $A$ and $B$, we have the following comparison between the various tensor norms:

$$
\| \cdot \|_\lambda \leq \| \cdot \|_h \leq \| \cdot \|_\wedge \leq \| \cdot \|_\gamma.
$$
Properties

- **commutative**: $V_1 \hat{\otimes} V_2 \overset{cb}{=} V_2 \hat{\otimes} V_1$;

- **associative**: $V_1 \hat{\otimes} (V_2 \hat{\otimes} V_3) \overset{cb}{=} (V_1 \hat{\otimes} V_2) \hat{\otimes} V_3$;

- **functorial**: If $\phi_i : V_i \to W_i, i = 1,2$ are completely bounded maps, then $\phi_1 \hat{\otimes} \phi_2 : V_1 \hat{\otimes} V_2 \to W_1 \hat{\otimes} W_2$ extends to a completely bounded map $\phi_1 \hat{\otimes} \phi_2 : V_1 \hat{\otimes} V_2 \to W_1 \hat{\otimes} W_2$ with $\|\phi_1 \hat{\otimes} \phi_2\|_{cb} \leq \|\phi_1\|_{cb} \|\phi_2\|_{cb}$;

- **projective**: For closed subspaces $W_i \subseteq V_i, i = 1,2$, the tensor map $V_1 \otimes V_2 \to (V_1/W_1) \otimes (V_2/W_2)$ extends to a complete quotient map $V_1 \hat{\otimes} V_2 \to (V_1/W_1) \hat{\otimes} (V_2/W_2)$;
- Not injective, that is, for subspaces \( W_i \subseteq V_i, i = 1, 2 \), the induced map \( W_1 \hat{\otimes} W_2 \rightarrow V_1 \hat{\otimes} V_2 \) is not a complete isometry.

- (Effros and Ruan, 1991) For \( W_1 \hookrightarrow W_2 \), the induced map \( V \hat{\otimes} W_1 \hookrightarrow V \hat{\otimes} W_2 \) is a complete isometry if and only if the dual operator space \( V^* \) is injective (that is for any operator spaces \( W_0 \subseteq W \), every c.b. linear map \( \phi_0 : W_0 \rightarrow V^* \) has a c.b. norm preserving linear extension \( \phi : W \rightarrow V^* \)).
• Not injective, that is, for subspaces $W_i \subseteq V_i$, $i = 1, 2$, the induced map $W_1 \hat{\otimes} W_2 \rightarrow V_1 \hat{\otimes} V_2$ is not a complete isometry.

• (Effros and Ruan, 1991) For $W_1 \hookrightarrow W_2$, the induced map $V \hat{\otimes} W_1 \hookrightarrow V \hat{\otimes} W_2$ is a complete isometry if and only if the dual operator space $V^*$ is injective (that is for any operator spaces $W_0 \subseteq W$, every c.b. linear map $\phi_0 : W_0 \rightarrow V^*$ has a c.b. norm preserving linear extension $\phi : W \rightarrow V^*$.)
Given a certain tensor norm \( \alpha \), and spaces \( A, B \), is it possible to embed \( A^{**} \otimes^\alpha B^{**} \) into \((A \otimes^\alpha B)^{**}\) isometrically (completely), or, at least bi-continuously?

(Archbold and Batty, 1980) A \( C^* \)-algebra \( A \) is said to satisfy Property C if for every \( C^* \)-algebra \( B \), the \( C^* \)-algebra \( A^{**} \otimes_{\text{min}} B^{**} \) can be isometrically embedded in \((A \otimes_{\text{min}} B)^{**}\).

For Banach spaces \( E \) and \( F \), the inclusion

\[
E^{**} \otimes F^{**} \hookrightarrow (E \otimes_{\lambda} F)^{**}
\]

induces the Banach space injective tensor norm \( \| \cdot \|_{\lambda} \) on \( E^{**} \otimes F^{**} \).

If \( V \) and \( W \) are reflexive operator spaces and \( \alpha \) is an operator space tensor norm, then \( V^{**} \otimes_{\alpha} W^{**} \) can be completely isometrically embedded in \((V \otimes_{\alpha} W)^{**}\).
(Effros and Ruan) For operator spaces $V$ and $W$, there is a separately $w^*$-continuous extension $\theta : V^{**} \otimes W^{**} \rightarrow (V \tilde{\otimes} W)^{**}$ of the natural inclusion $i : V \otimes W \rightarrow (V \tilde{\otimes} W)^{**}$, which is also injective. An operator space $V$ is said to satisfy condition $C$ if for every operator space $W$, the map $\theta$ is (completely) isometric with respect to the injective norm.

(Kumar and Sinclair, 1998) For $C^*$-algebras $A$ and $B$, the canonical embedding $\mu$ of $A^{**} \otimes^\gamma B^{**}$ into $(A \otimes^\gamma B)^{**}$ satisfies

$$\|u\| \leq 4\|\mu(u)\| \leq 4\|u\|$$

for all $u \in A^{**} \otimes^\gamma B^{**}$. In particular, $\mu$ has a continuous inverse.
Theorem (Jain and Kumar)

For exact operator spaces $V$ and $W$, there is a canonical embedding $\mu$ of $V^{**} \hat{\otimes} W^{**}$ into $(V \hat{\otimes} W)^{**}$ which satisfies

$$\frac{1}{2K} ||u|| \leq ||\mu(u)|| \leq 2K ||u|| \text{ for all } u \in V^{**} \hat{\otimes} W^{**},$$

where $K = 2\sqrt{2} \text{ex}(V) \text{ex}(W)$. In particular, $\mu$ has a continuous inverse.

An operator space $V$ is exact if

$$\{0\} \rightarrow K(\ell_2) \hat{\otimes} V \rightarrow B(\ell_2) \hat{\otimes} V \rightarrow \frac{B(\ell_2)}{K(\ell_2)} \hat{\otimes} V \rightarrow \{0\}$$

is $1$-exact. The exactness constant of $V$ is defined as $\text{ex}(V) = \|T_V^{-1}\|$, where

$$T_V : \frac{B(\ell_2) \hat{\otimes} V}{K(\ell_2) \hat{\otimes} V} \rightarrow \frac{B(\ell_2)}{K(\ell_2)} \hat{\otimes} V.$$
Embeddings of tensor products into bidual

Theorem (___ and Kumar)

For exact operator spaces $V$ and $W$, there is a canonical embedding $\mu$ of $V^{**} \hat{\otimes} W^{**}$ into $(V \hat{\otimes} W)^{**}$ which satisfies

$$\frac{1}{2K} \|u\| \leq \|\mu(u)\| \leq 2K \|u\| \text{ for all } u \in V^{**} \hat{\otimes} W^{**},$$

where $K = 2\sqrt{2} \text{ex}(V)\text{ex}(W)$. In particular, $\mu$ has a continuous inverse.

- An operator space $V$ is exact if

$$\{0\} \rightarrow \mathcal{K}(\ell_2) \hat{\otimes} V \rightarrow \mathcal{B}(\ell_2) \hat{\otimes} V \rightarrow \frac{\mathcal{B}(\ell_2)}{\mathcal{K}(\ell_2)} \hat{\otimes} V \rightarrow \{0\}$$

is 1-exact. The exactness constant of $V$ is defined as

$$\text{ex}(V) = \|T_V^{-1}\|,$$

where

$$T_V : \frac{\mathcal{B}(\ell_2)}{\mathcal{K}(\ell_2)} \hat{\otimes} V \rightarrow \frac{\mathcal{B}(\ell_2)}{\mathcal{K}(\ell_2)} \hat{\otimes} V.$$
Outline of Proof

- Every j.c.b. bilinear map \( u : V \times W \to \mathbb{C} \) can be extended uniquely to a separately \( w^* \)-continuous j.c.b. bilinear map \( \tilde{u} : V^{**} \times W^{**} \to \mathbb{C} \) such that \( \| \tilde{u} \|_{jcb} \leq 2K \| u \|_{jcb} \), where \( K = 2\sqrt{2} \text{ex}(V)\text{ex}(W) \).

- We have a map \( \chi : (V \hat{\otimes} W)^* \to (V^{**} \hat{\otimes} W^{**})^* \) with \( \| \chi \| \leq 2K \). Define
  \[
  \mu := \chi^* \circ i : V^{**} \hat{\otimes} W^{**} \to (V \hat{\otimes} W)^{**},
  \]
  where \( i : V^{**} \hat{\otimes} W^{**} \to (V^{**} \hat{\otimes} W^{**})^{**} \) is the canonical completely isometric embedding.

Theorem (___ and Kumar)

For \( C^* \)-algebras \( A \) and \( B \), there is a canonical bicontinuous embedding \( \mu \) of \( A^{**} \hat{\otimes} B^{**} \) into \( (A \hat{\otimes} B)^{**} \) satisfying

\[
\frac{1}{2} \| u \| \leq \| \mu(u) \| \leq \| u \| \quad \text{for all } u \in A^{**} \hat{\otimes} B^{**}.
\]
Theorem (___ and Kumar)

For operator spaces $V$ and $W$, there is a canonical completely isometric embedding of $V^{**} \otimes^h W^{**}$ into $(V \otimes^h W)^{**}$.

Outline of the proof:

- Let $E$ be a dual operator space and $u : V \times W \to E$ be a c.b. bilinear map. Then $u$ admits a unique separately $w^*$-continuous c.b. extension $\tilde{u} : V^{**} \times W^{**} \to E$, with $\|u\|_{cb} = \|	ilde{u}\|_{cb}$.

- This leads to prove that $(V \otimes^h W)^*$ is completely isometrically isomorphic to $(V^{**} \otimes^h W^{**})^*$, so that $(V \otimes^h W)^{**}$ becomes completely isometrically isomorphic to $V^{**} \otimes_{\sigma h} W^{**}$, the normal Haagerup tensor product.

- The result now follows from the fact that there is a complete isometric embedding $V^{**} \otimes^h W^{**} \hookrightarrow V^{**} \otimes_{\sigma h} W^{**}$. 
For $C^*$-algebras $A$ and $B$, the Haagerup norm is equivalent to the operator space projective tensor norm on $A \otimes B$ if and only if either $A$ or $B$ is finite dimensional, or $A$ and $B$ are infinite dimensional and subhomogeneous. 

(A $C^*$-algebra $A$ is said to be subhomogeneous if every irreducible $*$-representation of $A$ has dimension less than or equal to $n$, for some $n \in \mathbb{N}$.)
(Dimant and Unzeuta, 2016) Let $V$ and $W$ be operator spaces with $V^*$ and $W^*$ as locally reflexive. If either $V^{**}$ or $W^{**}$ has CBAP (completely bounded approximation property), then there is a natural embedding of $V^{**} \hat{\otimes} W^{**} \hookrightarrow (V \hat{\otimes} W)^{**}$ which is complete isometry.

For locally reflexive operator spaces $V$ and $W$, the natural embedding of $V^{**} \check{\otimes} W^{**} \hookrightarrow (V \check{\otimes} W)^{**}$ is complete isometry.

Obtained analogous statement for finitely generated operator space tensor norm.

V. Dimant and M. F. Unzeuta, Biduals of Tensor Products in Operator Spaces, Studia Mathematica, 2016.
(Kumar, 2001) For $C^*$-algebras $A$ and $B$, $A \hat{\otimes} B$ is a Banach*-algebra.

(Kumar, 2001) If $I$ and $J$ are closed ideals of $A$ and $B$, then $I \hat{\otimes} J$ is a closed $\ast$-ideal of $A \hat{\otimes} B$.

(--- and Kumar) Let $I_1, I_2$ and $J_1, J_2$ be the closed ideals of $A$ and $B$ respectively, where $A$ and $B$ are $C^*$-algebras. Then $I_1 \hat{\otimes} J_1 + I_2 \hat{\otimes} J_2$ is a closed $\ast$-ideal of $A \hat{\otimes} B$.

(--- and Kumar) For $C^*$-algebras $A$ and $B$, the canonical map $i : A \hat{\otimes} B \to A \otimes^{\min} B$ is injective.

The Banach $\ast$-algebra $A \hat{\otimes} B$ is simple if and only if $A$ and $B$ are both simple.

Algebraic structure of $A \hat{\otimes} B$

- (Kumar, 2001) For $C^*$-algebras $A$ and $B$, $A \hat{\otimes} B$ is a Banach*-algebra.

- (Kumar, 2001) If $I$ and $J$ are closed ideals of $A$ and $B$, then $I \hat{\otimes} J$ is a closed $*$-ideal of $A \hat{\otimes} B$.

- (__) and Kumar) Let $I_1, I_2$ and $J_1, J_2$ be the closed ideals of $A$ and $B$ respectively, where $A$ and $B$ are $C^*$-algebras. Then $I_1 \hat{\otimes} J_1 + I_2 \hat{\otimes} J_2$ is a closed $*$-ideal of $A \hat{\otimes} B$.

- (__) and Kumar) For $C^*$-algebras $A$ and $B$, the canonical map $i : A \hat{\otimes} B \to A \otimes_{\min} B$ is injective.

- The Banach $*$-algebra $A \hat{\otimes} B$ is simple if and only if $A$ and $B$ are both simple.

For a simple $C^*$-algebra $A$, every closed ideal of $A\hat{\otimes}B$ has the form $A\hat{\otimes}L$ for some closed ideal $L$ of $B$.

A closed ideal $K$ of $A\hat{\otimes}B$ is minimal if and only if there exist minimal closed ideals $I$ and $J$ of $A$ and $B$, respectively, such that $K = I\hat{\otimes}J$.

**Theorem (___ and Kumar)**

A closed ideal $K$ of $A\hat{\otimes}B$ is maximal if and only if there exist maximal closed ideals $M$ and $N$ of $A$ and $B$ respectively, such that

$$K = A\hat{\otimes}N + M\hat{\otimes}B.$$
For each $\phi \in A^*$, define a linear map $R_\phi : A \otimes B \to B$ by

$$R_\phi(\sum_{i=1}^n a_i \otimes b_i) = \sum_{i=1}^n \phi(a_i)b_i.$$  

The map $R_\phi$ is continuous with respect to the operator space projective tensor norm with $\|R_\phi\| \leq \|\phi\|$; thus, it can be extended to $A \hat{\otimes} B$ as a bounded linear map. This map is known as the right slice map associated to $\phi$.

For a closed ideal $J$ of $B$, $R_\phi(x) \in J$ for all $\phi \in A^*$, and for every $x \in A \hat{\otimes} J$.

A triple $(A, B, J)$, where $J$ is a closed two-sided ideal of $B$, is said to verify the slice map conjecture if whenever $x \in A \otimes B$ and $R_\phi(x) \in J$ for all $\phi \in A^*$ then $x \in A \otimes J$.

Slice map conjecture is not true for min-norm on $C^*$-algebras.
Smith, 1988, proved it for all subspaces $J$ for Haagerup tensor product of $C^*$-algebras.

(_, and Kumar) Slice map conjecture is true for operator space projective tensor product of $C^*$-algebras.

Let $I_1$, $I_2$ and $J_1$, $J_2$ be closed ideals of $A$ and $B$, respectively. Then,

$$(I_1 \hat{\otimes} J_1) \cap (I_2 \hat{\otimes} J_2) = (I_1 \cap I_2) \hat{\otimes} (J_1 \cap J_2).$$
For a closed ideal \( K \) of \( \hat{\bigotimes}B \), \( K \) is prime if and only if there exist some closed prime ideals \( E \) and \( F \) of \( A \) and \( B \), respectively, such that \( K = A\hat{\bigotimes}F + E\hat{\bigotimes}B \).

Outline of Proof:

- For a closed ideal \( K \), choose closed ideals \( E \) and \( F \) of \( A \) and \( B \) which are maximal w.r.t. the fact that \( A\hat{\bigotimes}F + E\hat{\bigotimes}B \subseteq K \).

- For quotient maps \( \pi : A \to A/E \) and \( \rho : B \to B/F \), slice map property gives \((\pi\hat{\bigotimes}\rho)(K) = 0\) so that \( K = A\hat{\bigotimes}F + E\hat{\bigotimes}B \).

- Also, if \( I \cap J \subseteq E \), then \( (I\hat{\bigotimes}B) \cap (J\hat{\bigotimes}B) \subseteq K \) so either \( I\hat{\bigotimes}B \subseteq K \) or \( J\hat{\bigotimes}B \in K \). which further gives \( I \subseteq E \) or \( J \subseteq E \).
Theorem (— and Kumar)

For $C^*$-algebras $A$ and $B$, we have the following:

(i) If $E$ and $F$ are primitive ideals of $A$ and $B$ respectively, then $A \hat{\otimes} F + E \hat{\otimes} B$ is a primitive ideal of $A \hat{\otimes} B$.

(ii) If $K$ is a primitive ideal of $A \hat{\otimes} B$, then $K = A \hat{\otimes} F + E \hat{\otimes} B$ for some closed prime ideals $E$ and $F$ of $A$ and $B$, respectively.

(iii) If $A$ and $B$ are separable, then an ideal $K$ of $A \hat{\otimes} B$ is primitive if and only if $K = A \hat{\otimes} F + E \hat{\otimes} B$ for some primitive ideals $E$ and $F$ of $A$ and $B$, respectively.
Lattice of closed ideals of $\mathcal{B}(H) \widehat{\otimes} \mathcal{B}(H)$

\[
\begin{align*}
&\mathcal{B}(H) \widehat{\otimes} \mathcal{B}(H) \\
\downarrow & \\
&\mathcal{B}(H) \widehat{\otimes} \mathcal{K}(H) + \mathcal{K}(H) \widehat{\otimes} \mathcal{B}(H) \\
\downarrow & \\
&\mathcal{B}(H) \widehat{\otimes} \mathcal{K}(H) & \text{or} & \mathcal{K}(H) \widehat{\otimes} \mathcal{B}(H) \\
\downarrow & \\
&\mathcal{K}(H) \widehat{\otimes} \mathcal{K}(H) \\
\downarrow & \\
&\{0\}
\end{align*}
\]
Thank you for your attention!