Operator Space Projective Tensor Product: Embedding into second dual and ideal structures

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Overview

- Operator Spaces
- Tensor Products
- Operator space projective tensor product
- Embeddings of tensor products into bidual
- Algebraic structure

Introduction- Operator Space

- An (concrete) operator space V is a subspace of $\mathcal{B}(H)$ together with the natural norms on $M_n(V)$ inherited from $M_n(\mathcal{B}(H)) = \mathcal{B}(H^n)$, H being a Hilbert space.
- Every Banach space possesses an operator space structure.To see this, for $\Gamma = B_1(X^*)$, the isometric embedding $X \hookrightarrow C(\Gamma)$ via $x \to f_x$, where $f_x(g) := g(x)$ for $g \in C(\Gamma)$, equips X with an operator space structure.
- Every C^*-algebra is an operator space.
- For Hilbert spaces H and K, $\mathcal{B}(H, K) \subseteq \mathcal{B}(H \oplus K)$ is an operator space.

Introduction- Operator Space (Morphisms)

- A linear map $\phi : V \to W$ between operator spaces V and W is said to be *completely bounded* (in short, c.b.) if

$$\|\phi\|_{cb} := \sup\{\|\phi_n\| : n \in \mathbb{N}\} < \infty,$$

where $\phi_n : M_n(V) \to M_n(W)$ is defined by

$$\phi_n((x_{ij})) = (\phi(x_{ij})) \text{ for all } (x_{ij}) \in M_n(V).$$

- Two operator spaces V and W are said to be *completely isomorphic* if there exists a completely bounded linear bijection $\phi : V \to W$ whose inverse is also completely bounded.

Introduction- Operator Space-Abstract Characterization

Definition:

A normed space V with a sequence of norms

$$\|\cdot\|_n : M_n(V) \to [0,\infty), \quad n \in \mathbb{N}$$

is said to be an *(abstract) operator space* if:

(i) $\|\begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}\|_{n+m} = \max\{\|v\|_n, \|w\|_m\}$ for all $v \in M_n(V), w \in M_m(V)$

(ii) $\|\alpha v \beta\|_m \leq \|\alpha\| \, \|v\|_n \|\beta\|$, for all $\alpha \in \mathbb{M}_{m,n}, \beta \in \mathbb{M}_{n,m}, v \in M_n(V)$.

Ruan, 1988

If V is an abstract operator space, then V is completely isometrically isomorphic to a linear subspace of $\mathcal{B}(H)$ for some Hilbert space H.

Z.-J Ruan, Subspace of C^*-algebras. J. Funct. Anal. (76) 1988

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Some Elementary Constructions

- (*c.b. maps*) $\mathcal{CB}(V,W)$, the space of completely bounded maps from V into W is an operator space with the matricial norms inherited from the identifications

$$M_n(\mathcal{CB}(V,W)) \cong \mathcal{CB}(V, M_n(W))$$

- (*Dual*) The operator space structure on the dual V^* comes by identifying $V^* = \mathcal{CB}(V,\mathbb{C}) \cong \mathcal{B}(V,\mathbb{C})$. Thus, V^* obtains its matricial structure from the identifications

$$M_n(V^*) \cong \mathcal{CB}(V, M_n(\mathbb{C})),$$

and the operator space thus obtained is known as the *operator space dual* of V.

Tensor Norms/Products

- For $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$, *Banach space injective tensor norm* is defined as

$$\|u\|_\lambda = \sup \Big\{ \Big| \sum_{i=1}^{n} f(x_i) g(y_i) \Big| : f \in B_1(X^*), g \in B_1(Y^*) \Big\}.$$

- Let X and Y be Banach spaces. For $u \in X \otimes Y$, *Banach space projective tensor norm* is defined as

$$\|u\|_\gamma = \inf \Big\{ \sum_{i=1}^{n} \|x_i\| \|y_i\| \, : \, u = \sum_{i=1}^{n} x_i \otimes y_i \Big\},$$

A. Grothendieck, *Products tensoriels topologiques et espaces nuclearies*. Memoirs of the American Mathematical Society (16), 1955.

- For a bilinear map $u : V \times W \to E$, there are two ways to look at its amplifications

  $u_n : M_n(V) \times M_n(W) \to M_n(E)$

$$u_n\big((v_{ij}),(w_{kl})\big) = \Big(\sum_{k=1}^{n} u(v_{ik}, w_{kj})\Big),\ n \in \mathbb{N}$$

  $u_n^{'} : M_n(V) \times M_n(W) \to M_{n^2}(E)$

$$u_n^{'}\big((v_{ij}),(w_{kl})\big) = \big(u(v_{ij}, w_{kl})\big),\ n \in \mathbb{N}$$

- u is completely bilinear (c.b.) if u_n are uniformly bounded, and we denote $\|u\|_{cb} = \sup_n \|u_n\|$.
- u is jointly completely bilinear (j.c.b.) if $u_n^{'}$ are uniformly bounded, and we denote $\|u\|_{jcb} = \sup_n \|u_n^{'}\|$.

J.C.B. Bilinear Map

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Haagerup Tensor Product

Definition

For operator spaces V and W, the *Haagerup tensor norm* is defined as

$$\|u\|_h = \inf\{\|v\|\|w\| : u = v \odot w, v \in M_{n,p}(V), w \in M_{p,n}(W), p \in \mathbb{N}\},$$

where $u \in M_n(V \otimes W)$ and $v \odot w = \left(\sum_{k=1}^{p} v_{ik} \otimes w_{kj}\right)_{ij}$.

- For C^*-algebras A and B, the Haagerup norm of an element $u \in A \otimes B$ takes a simpler and convenient form given by

$$\|u\|_h = \inf\left\{\|\Sigma_i a_i a_i^*\|^{1/2} \|\Sigma_i b_i^* b_i\|^{1/2} : u = \Sigma_{i=1}^{n} a_i \otimes b_i\right\}.$$

- $(V \otimes^h W)^* \overset{cb}{=} \text{CB}(V \times W, \mathbb{C})$.

Effros and Kishimoto
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Operator space projective tensor product

Definition

The *operator space projective tensor product* denoted by $V\widehat{\otimes}W$, is the completion of the algebraic tensor product $V \otimes W$ under the norm

$$\|u\|_{\wedge} = \inf\{\|\alpha\|\|v\|\|w\|\|\beta\| : u = \alpha(v \otimes w)\beta\},\ u \in M_n(V \otimes W),$$

where infimum runs over arbitrary decompositions with $v \in M_p(V)$, $w \in M_q(W), \alpha \in M_{n,pq}, \beta \in M_{pq,n}$ and $p, q \in \mathbb{N}$ arbitrary.

- $(V\widehat{\otimes}W)^* \stackrel{cb}{=} \mathrm{JCB}(V \times W, \mathbb{C}) \stackrel{cb}{=} \mathcal{CB}(V, W^*) \stackrel{cb}{=} \mathcal{CB}(W, V^*)$

D. Blecher and V. Paulsen, Tensor product of operator spaces. J. Funct. Anal. (99) 1991.
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Comparison

On the tensor product $A \otimes B$ of two C^*-algebras A and B, we have the following comparison between the various tensor norms:

$$\| \cdot \|_\lambda \leq \| \cdot \|_h \leq \| \cdot \|_\wedge \leq \| \cdot \|_\gamma .$$

Properties

- *commutative*: $V_1 \widehat{\otimes} V_2 \stackrel{cb}{=} V_2 \widehat{\otimes} V_1$;
- *associative*: $V_1 \widehat{\otimes} (V_2 \widehat{\otimes} V_3) \stackrel{cb}{=} (V_1 \widehat{\otimes} V_2) \widehat{\otimes} V_3$;
- *functorial*: If $\phi_i : V_i \to W_i, i = 1, 2$ are completely bounded maps, then $\phi_1 \otimes \phi_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$ extends to a completely bounded map $\phi_1 \widehat{\otimes} \phi_2 : V_1 \widehat{\otimes} V_2 \to W_1 \widehat{\otimes} W_2$ with $\|\phi_1 \widehat{\otimes} \phi_2\|_{\mathrm{cb}} \leq \|\phi_1\|_{\mathrm{cb}} \|\phi_2\|_{\mathrm{cb}}$;
- *projective*: For closed subspaces $W_i \subseteq V_i, i = 1, 2$, the tensor map $V_1 \otimes V_2 \to (V_1/W_1) \otimes (V_2/W_2)$ extends to a complete quotient map $V_1 \widehat{\otimes} V_2 \to (V_1/W_1) \widehat{\otimes} (V_2/W_2)$;

Drawback

- Not injective, that is, for subspaces $W_i \subseteq V_i, i = 1, 2$, the induced map $W_1 \widehat{\otimes} W_2 \to V_1 \widehat{\otimes} V_2$ is not a complete isometry.

- (Effros and Ruan, 1991) For $W_1 \hookrightarrow W_2$, the induced map $V \widehat{\otimes} W_1 \hookrightarrow V \widehat{\otimes} W_2$ is a complete isometry if and only if the dual operator space V^* is injective (that is for any operator spaces $W_0 \subseteq W$, every c.b. linear map $\phi_0 : W_0 \to V^*$ has a c.b. norm preserving linear extension $\phi : W \to V^*$.)

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Embeddings of tensor products into bidual

- Given a certain tensor norm α, and spaces A, B, is it possible to embed $A^{**} \otimes^{\alpha} B^{**}$ into $(A \otimes^{\alpha} B)^{**}$ isometrically (completely), or, at least bi-continuously?
- (Archbold and Batty, 1980) A C^*-algebra A is said to satisfy *Property C* if for every C^*-algebra B, the C^*-algebra $A^{**} \otimes^{\min} B^{**}$ can be isometrically embedded in $(A \otimes^{\min} B)^{**}$.
- For Banach spaces E and F, the inclusion

  $$E^{**} \otimes F^{**} \hookrightarrow (E \otimes_{\lambda} F)^{**}$$

  induces the Banach space injective tensor norm $\| \cdot \|_{\lambda}$ on $E^{**} \otimes F^{**}$
- If V and W are reflexive operator spaces and α is an operator space tensor norm, then $V^{**} \otimes_{\alpha} W^{**}$ can be completely ismoetrically embedded in $(V \otimes_{\alpha} W)^{**}$.

Embeddings of tensor products into bidual

- (Effros and Ruan) For operator spaces V and W, there is a separately w^*-continuous extension $\theta : V^{**} \otimes W^{**} \to (V \check{\otimes} W)^{**}$ of the natural inclusion $i : V \otimes W \to (V \check{\otimes} W)^{**}$, which is also injective. An operator space V is said to satisfy *condition C* if for every operator space W, the map θ is (completely) isometric with respect to the injective norm.
- (Kumar and Sinclair, 1998) For C^*-algebras A and B, the canonical embedding μ of $A^{**} \otimes^\gamma B^{**}$ into $(A \otimes^\gamma B)^{**}$ satisfies

$$\|u\| \leq 4\|\mu(u)\| \leq 4\|u\|$$

for all $u \in A^{**} \otimes^\gamma B^{**}$. In particular, μ has a continuous inverse.

Embeddings of tensor products into bidual

Theorem (__ and Kumar)

For exact operator spaces V and W, there is a canonical embedding μ of $V^{**}\widehat{\otimes}W^{**}$ into $(V\widehat{\otimes}W)^{**}$ which satisfies

$$\frac{1}{2K}\|u\| \leq \|\mu(u)\| \leq 2K\|u\| \text{ for all } u \in V^{**}\widehat{\otimes}W^{**},$$

where $K = 2\sqrt{2}\, ex(V)ex(W)$. In particular, μ has a continuous inverse.

- An operator space V is *exact* if

$$\{0\} \longrightarrow \mathcal{K}(\ell_2)\check{\otimes}V \longrightarrow \mathcal{B}(\ell_2)\check{\otimes}V \longrightarrow \frac{\mathcal{B}(\ell_2)}{\mathcal{K}(\ell_2)}\check{\otimes}V \longrightarrow \{0\}$$

is 1-exact. The exactness constant of V is defined as $ex(V) = \|T_V^{-1}\|$, where

$$T_V : \frac{\mathcal{B}(\ell_2)\check{\otimes}V}{\mathcal{K}(\ell_2)\check{\otimes}V} \to \frac{\mathcal{B}(\ell_2)}{\mathcal{K}(\ell_2)}\check{\otimes}V.$$

R. Jain and A. Kumar. Operator space projective tensor product: Embedding into second dual and ideal structure. *Proc. Edin. Math. Soc.* **57** (2014)

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Outline of Proof

- Every j.c.b. bilinear map $u : V \times W \to \mathbb{C}$ can be extended uniquely to a separately w^*-continuous j.c.b. bilinear map $\tilde{u} : V^{**} \times W^{**} \to \mathbb{C}$ such that $\|\tilde{u}\|_{jcb} \leq 2K\|u\|_{jcb}$, where $K = 2\sqrt{2}\, ex(V)ex(W)$.

- We have a map $\chi : (V\widehat{\otimes}W)^* \to (V^{**}\widehat{\otimes}W^{**})^*$ with $\|\chi\| \leq 2K$. Define

$$\mu := \chi^* \circ i : V^{**}\widehat{\otimes}W^{**} \to (V\widehat{\otimes}W)^{**},$$

where $i : V^{**}\widehat{\otimes}W^{**} \to (V^{**}\widehat{\otimes}W^{**})^{**}$ is the canonical completely isometric embedding.

Theorem (__ and Kumar)

For C^*-algebras A and B, there is a canonical bicontinuous embedding μ of $A^{**}\widehat{\otimes}B^{**}$ into $(A\widehat{\otimes}B)^{**}$ satisfying

$$\frac{1}{2}\|u\| \leq \|\mu(u)\| \leq \|u\| \quad \textit{for all } u \in A^{**}\widehat{\otimes}B^{**}.$$

Embeddings of tensor products into bidual

Theorem (__ and Kumar)

For operator spaces V and W, there is a canonical completely isometric embedding of $V^{**} \otimes^h W^{**}$ into $(V \otimes^h W)^{**}$.

Outline of the proof:

- Let E be a dual operator space and $u : V \times W \to E$ be a c.b. bilinear map. Then u admits a unique separately w^*-continuous c.b. extension $\tilde{u} : V^{**} \times W^{**} \to E$, with $\|u\|_{\mathsf{cb}} = \|\tilde{u}\|_{\mathsf{cb}}$.
- This leads to prove that $(V \otimes^h W)^*$ is completely isometrically isomorphic to $(V^{**} \otimes^h W^{**})^*_\sigma$, so that $(V \otimes^h W)^{**}$ becomes completely isometrically isomorphic to $V^{**} \otimes_{\sigma h} W^{**}$, the normal Haagerup tensor product.
- The result now follows from the fact that there is a complete isometric embedding $V^{**} \otimes^h W^{**} \hookrightarrow V^{**} \otimes_{\sigma h} W^{**}$.

Consequence

- For C^*-algebras A and B, the Haagerup norm is equivalent to the operator space projective tensor norm on $A \otimes B$ if and only if either A or B is finite dimensional, or A and B are infinite dimensional and subhomogeneous.
  (A C^*-algebra A is said to be *subhomogeneous* if every irreducible $*$-representation of A has dimension less than or equal to n, for some $n \in \mathbb{N}$.)

Embeddings of tensor products into bidual

- (Dimant and Unzeuta, 2016) Let V and W be operator spaces with V^* and W^* as locally reflexive. If either V^{**} or W^{**} has CBAP (completely bounded approximation property), then there is a natural embedding of $V^{**}\widehat{\otimes}W^{**} \hookrightarrow (V\widehat{\otimes}W)^{**}$ which is complete isometry.
- For locally reflexive operator spaces V and W, the natural embedding of $V^{**}\check{\otimes}W^{**} \hookrightarrow (V\check{\otimes}W)^{**}$ is complete isometry.
- Obtained analogous statement for finitely generated operator space tensor norm.

V. Dimant and M. F. Unzeuta, Biduals of Tensor Products in Operator Spaces, Studia Mathematica, 2016.

- (Kumar, 2001) For C^*-algebras A and B, $A\widehat{\otimes}B$ is a Banach*-algebra.
- (Kumar, 2001) If I and J are closed ideals of A and B, then $I\widehat{\otimes}J$ is a closed $*$-ideal of $A\widehat{\otimes}B$.
- (__ and Kumar) Let I_1, I_2 and J_1, J_2 be the closed ideals of A and B respectively, where A and B are C^*-algebras. Then $I_1\widehat{\otimes}J_1 + I_2\widehat{\otimes}J_2$ is a closed $*$-ideal of $A\widehat{\otimes}B$.
- (__ and Kumar) For C^*-algebras A and B, the canonical map $i : A\widehat{\otimes}B \to A \otimes^{\min} B$ is injective.
- The Banach $*$-algebra $A\widehat{\otimes}B$ is simple if and only if A and B are both simple.

1. S. D. Allen, A. M. Sinclair and R. R. Smith. The ideal structure of the Haagerup tensor product of C^*-algebras. *J. Reine Angew. Math.* **442** (1993)
2. R. Jain and A. Kumar. Ideals of operator space projective tensor product of C^*-algebras. *J. Aust. Math. Soc.* **91** (2011)

Algebraic structure of $A\widehat{\otimes}B$

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Algebraic structure of $A\widehat{\otimes}B$

- For a simple C^*-algebra A, every closed ideal of $A\widehat{\otimes}B$ has the form $A\widehat{\otimes}L$ for some closed ideal L of B.
- A closed ideal K of $A\widehat{\otimes}B$ is minimal if and only if there exist minimal closed ideals I and J of A and B, respectively, such that $K = I\widehat{\otimes}J$.

Theorem (__ and Kumar)

A closed ideal K of $A\widehat{\otimes}B$ is maximal if and only if there exist maximal closed ideals M and N of A and B respectively, such that

$$K = A\widehat{\otimes}N + M\widehat{\otimes}B.$$

Algebraic structure of $A\widehat{\otimes}B$- Slice map

- For each $\phi \in A^*$, define a linear map $R_\phi : A \otimes B \to B$ by

$$R_\phi(\Sigma_{i=1}^n a_i \otimes b_i) = \Sigma_{i=1}^n \phi(a_i)b_i.$$

  The map R_ϕ is continuous with respect to the operator space projective tensor norm with $\|R_\phi\| \leq \|\phi\|$; thus, it can be extended to $A\widehat{\otimes}B$ as a bounded linear map. This map is known as the *right slice map* associated to ϕ.

- For a closed ideal J of B, $R_\phi(x) \in J$ for all $\phi \in A^*$, and for every $x \in A\widehat{\otimes}J$.

- A triple (A, B, J), where J is a closed two-sided ideal of B, is said to verify the *slice map conjecture* if whenever $x \in A \otimes B$ and $R_\phi(x) \in J$ for all $\phi \in A^*$ then $x \in A \otimes J$.

- Slice map conjecture is not true for min-norm on C^*-algebras.

Slice Map

- Smith, 1988, proved it for all subspaces J for Haagerup tensor product of C^*-algebras.
- (__ and Kumar) Slice map conjecture is true for operator space projective tensor product of C^*-algebras.
- Let I_1, I_2 and J_1, J_2 be closed ideals of A and B, respectively. Then,
$$(I_1 \widehat{\otimes} J_1) \cap (I_2 \widehat{\otimes} J_2) = (I_1 \cap I_2) \widehat{\otimes} (J_1 \cap J_2).$$

Theorem (__ and Kumar)

For a closed ideal K of $A\widehat{\otimes}B$, K is prime if and only it there exist some closed prime ideals E and F of A and B, respectively, such that $K = A\widehat{\otimes}F + E\widehat{\otimes}B$.

Outline of Proof:

- For a closed ideal K, choose closed ideals E and F of A and B which are maximal w.r.t. the fact that $A\widehat{\otimes}F + E\widehat{\otimes}B \subseteq K$.
- For quotient maps $\pi : A \to A/E$ and $\rho : B \to B/F$, slice map property gives $(\pi\widehat{\otimes}\rho)(K) = 0$ so that $K = A\widehat{\otimes}F + E\widehat{\otimes}B$.
- Also, if $I \cap J \subseteq E$, then $(I\widehat{\otimes}B) \cap (J\widehat{\otimes}B) \subseteq K$ so either $I\widehat{\otimes}B \subseteq K$ or $J\widehat{\otimes}B \in K$. which further gives $I \subseteq E$ or $J \subseteq E$.

Primitive Ideals

Theorem (__ and Kumar)

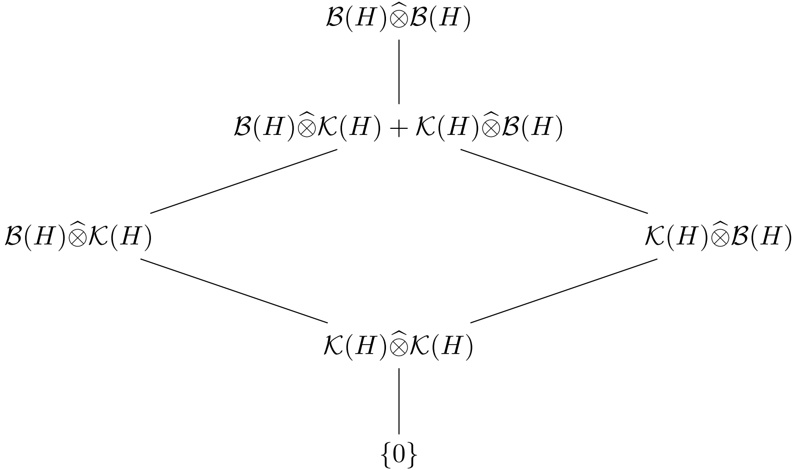
For C^*-algebras A and B, we have the following:

(i) *If E and F are primitive ideals of A and B respectively, then $A\widehat{\otimes}F + E\widehat{\otimes}B$ is a primitive ideal of $A\widehat{\otimes}B$.*

(ii) *If K is a primitive ideal of $A\widehat{\otimes}B$, then $K = A\widehat{\otimes}F + E\widehat{\otimes}B$ for some closed prime ideals E and F of A and B, respectively.*

(iii) *If A and B are separable, then an ideal K of $A\widehat{\otimes}B$ is primitive if and only if $K = A\widehat{\otimes}F + E\widehat{\otimes}B$ for some primitive ideals E and F of A and B, respectively.*

Lattice of closed ideals of $\mathcal{B}(H)\widehat{\otimes}\mathcal{B}(H)$



Thank you for your attention!