An intrinsic algebraic characterization of C*-simplicity (for discrete groups)

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August 17, 2016

C*-simplicity and the unique trace property

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Theorem (Powers 1975)

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We say that $\mathbb{F}_2$ is **C*-simple** and has the **unique trace property**.

Variants of Powers' proof became the main method for establishing these properties.

Definition

A group G has *Powers' averaging property* if for every $a \in \mathrm{C}^*_r(G)$ and $\epsilon > 0$ there are $g_1, \ldots, g_n \in G$ such that

$$\left\| \frac{1}{n} \sum \lambda_{g_i} a \lambda_{g_i^{-1}} - \tau(a)1 \right\| < \epsilon.$$

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Proof.

For C*-simplicity, let I be a non-trivial closed two-sided ideal of $\mathrm{C}^*_r(G)$. By faithfulness there is $a \in I$ with $\tau(a) = 1$. Applying Powers' averaging property implies $1 \in I$. The unique trace property is similarly straightforward.

□

Theorem (Powers 1975)

The free group $\mathbb{F}_2$ has Powers' averaging property. Hence it is C-simple and has the unique trace property.*

Question

Is there an (intrinsic) group-theoretic characterization of C*-simplicity and the unique trace property?

A group G is C*-simple iff whenever ρ is a unitary representation of G,

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C*-simple groups have no non-trivial normal amenable subgroups.

Proof.

If $N < G$ is amenable and normal then $\lambda_{G/N} \precnsim \lambda_G$. □

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C*-simple and unique trace equivalent to trivial amenable radical	**Authors**
Free groups $\mathbb{F}_n$ for $n \geq 2$	Powers (1975)
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Linear groups	T. Poznansky (2008)
Groups with non-zero first ℓ^2-Betti number	J. Peterson and A. Thom (2010)
Acylindrically hyperbolic groups	F. Dahmani, V. Guirardel, and D. Osin (2011)
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All the above results were proved using variants of Powers' ideas.

Problem

Are C*-simplicity and the unique trace property always equivalent to triviality of the amenable radical?

Characterizations of C*-simplicity and the unique trace property

Definition (Furstenberg 1973)

A compact G-space X is a G-**boundary** if for every probability measure $\mu \in \mathcal{P}(X)$, the weak* closure of the orbit $G\mu$ contains the point masses $\{\delta_x \mid x \in X\}$.

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Example

The Gromov boundary $\partial\mathbb{F}_n$ of the Free group $\mathbb{F}_n$ can be identified with the set of infinite reduced words

$$\partial\mathbb{F}_n = \{w = w_1 w_2 w_3 \cdots \mid w_i \in \{1, \dots, n\}\}.$$

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Fact

Every compact G-space contains a (potentially trivial) G-boundary. A trivial G-boundary is a fixed point.

Theorem (Kalantar-K 2014)

C-simplicity is equivalent to the existence of a (topologically) free action on a G-boundary.*

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Theorem (Breuillard-Kalantar-K-Ozawa 2014)

The unique trace property is equivalent to triviality of the amenable radical. In particular, every C-simple group has the unique trace property.*

A characterization of the unique trace property

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Specifically, tracial state on $\mathrm{C}_r^*(G)$ concentrate on the amenable radical $R_a(G)$, i.e. for every tracial state τ on $\mathrm{C}_r^*(G)$,

$$\tau(\lambda_s) = 0, \quad \forall s \in G \backslash R_a(G).$$

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Corollary

Every C-simple group has the unique trace property.*

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Example (Ivanov-Omland 2016)

Constructed from free products.

A new characterization of C*-simplicity

Let A be a unital G-C*-algebra. Then the state space $S(A)$, equipped with the weak* topology, is naturally a compact G-space.

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Theorem (K 2015)

A group G is C-simple if and only if the singleton $\{\tau\}$ is the only G-boundary in the state space $S(\mathrm{C}^*_\lambda(G))$.*

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For groups with the unique trace property that are not C*-simple, e.g. Le Boudec's examples, there is one singleton G-boundary (corresponding to the canonical trace) in $S(\mathrm{C}^*_r(G))$ and at least one non-singleton G-boundary in $S(\mathrm{C}^*_r(G))$.

Theorem (Haagerup 2015, K 2015)

A group G is C-simple if and only if it has Powers' averaging property, i.e. if and only if for every $a \in \mathrm{C}^*_r(G)$ and $\epsilon > 0$ there are $g_1, \ldots, g_n \in G$ such that*

$$\left\| \frac{1}{n} \sum \lambda_{g_i} a \lambda_{g_i^{-1}} - \tau(a)1 \right\| < \epsilon.$$

An (intrinsic) algebraic characterization of C*-simplicity

Let $\mathcal{S}(G)$ denote the space of subgroups of G, equipped with the Chabauty topology (i.e. the product topology on $\{0,1\}^G$).

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A sequence $(H_n) \subset \mathcal{S}(G)$ converges to $G \in \mathcal{S}(G)$ if

1. every $h \in H$ eventually belongs to H_n and
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Definition (Glasner-Weiss 2015)

A uniformly recurrent subgroup of G is a minimal (i.e. every orbit is dense) G-subspace of $S(G)$. It is amenable if it is a subset of the (closed) set of amenable subgroups of G.

Theorem (K 2015)

A group G is C-simple if and only if it has non-trivial amenable uniformly recurrent subgroups.*

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Key idea is that amenable uniformly recurrent subgroups correspond to boundaries in the state space of $\mathrm{C}^*_r(G)$.

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A subgroup $H < G$ is *recurrent* if for every sequence (g_n) in G there is a subsequence (g_{n_k}) such that

$$\bigcap g_{n_k} H g_{n_k}^{-1} \neq \{e\}.$$

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Theorem (K 2015)

A group G is C-simple if and only if it has no amenable recurrent subgroups.*

Connection to Thompson's groups

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The group F can be identified with the group of piecewise linear homeomorphisms of $[0, 1]$ that are differentiable, except at finitely many dyadic rationals, with derivative a power of 2 when it exists.

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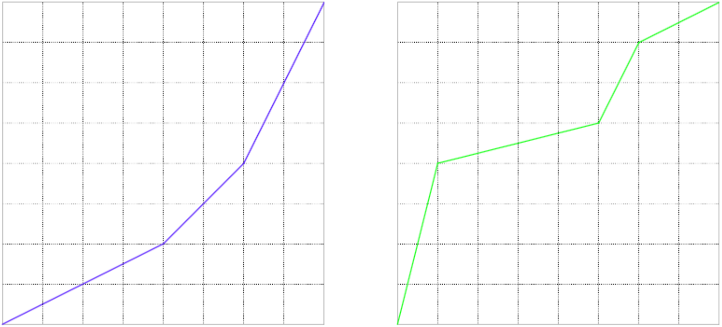

Big Open Question

Is F amenable?

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Theorem (Haagerup-Olesen 2014)

If T is C-simple, then F is non-amenable.*

Proof.

It is easy to check that F is a recurrent subgroup of T. If T is C*-simple, then it has no non-trivial amenable recurrent subgroups by [K2015]. Hence F is necessarily non-amenable. □

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Theorem (Le Boudec, Bon 2016)

1. *Every non-trivial recurrent subgroup of V is non-amenable.*
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Thompson's group V is C-simple.*

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Thompson's group F is non-amenable if and only if T is C-simple.*

Proof.

If F is non-amenable then every non-trivial recurrent subgroup of T (which contains a copy of F) is non-amenable. Hence T has no non-trivial amenable recurrent subgroups, so the result follows from [K2015]. □

Application to crossed products

Let (A, G, α, σ) be a (twisted) C*-dynamical system and let $I \lhd A$ be closed and G-invariant.

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0 & \longrightarrow & I \rtimes_{\alpha,r}^{\sigma} G & \longrightarrow & A \rtimes_{\alpha,r}^{\sigma} G & \xrightarrow{\pi \rtimes_{\alpha,r}^{\sigma} \mathrm{id}} & A/I \rtimes_{\dot{\alpha},r}^{\dot{\sigma}} G & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle E_I} & & \Big\downarrow {\scriptstyle E_A} & & \Big\downarrow {\scriptstyle E_{A/I}} & & \\
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We say (A, G, α, σ) is **exact** if the upper sequence is exact .This happens iff

$$I \rtimes^{\sigma}_{\alpha,r} G = \ker(\pi \rtimes^{\sigma}_{\alpha,r} \mathrm{id}) =: I \overline{\rtimes}^{\sigma}_{\alpha,r} G.$$

Theorem (Bedos-Conti 2015)

Let G be a "Powers-type" group and let (A, G, α, σ) be an exact twisted C-dynamical system. Then there is bijective correspondence between maximal closed ideals of $A \rtimes^{\sigma}_{r,\alpha} G$ and maximal G-invariant closed ideals of A.*

Theorem (Bryder-K 2016)

Let G be a C-simple group and let (A, G, α, σ) be a (not necessarily exact) twisted C*-dynamical system. Then there is bijective correspondence between maximal closed ideals of $A \rtimes_{r,\alpha}^{\sigma} G$ and maximal G-invariant closed ideals of A:*

$$A \rtimes_{r,\alpha}^{\sigma} G \rhd J \mapsto J \cap A \lhd A$$

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Corollary

Let (A, G, α, σ) be a twisted C-dynamical system over a C*-simple group Then $A \rtimes^{\sigma}_{r,\alpha} G$ is simple if and only if A has no proper non-trivial G-invariant ideals. In particular, if $A = C(X)$ then $C(X) \rtimes^{\sigma}_{r,\alpha} G$ is simple if and only if $G \curvearrowright X$ is minimal.*

Thanks!