The $C^*$-algebras of some exponential Lie groups

Göteborg, August 15 2016
Let $A$ be an involutive Banach algebra.
The $C^*$-algebra of $A$ is the completion $C^*(A)$ of $A$ with respect to
the $C^*$-norm

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\|a\|_{C^*} := \sup_{\pi \in \text{Rep}} \|\pi(a)\|_{\text{op}}
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where $\text{Rep}(A)$ denotes the collection of all unitary representation
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where

$$
\int_G \Delta_G(t^{-1})f(t^{-1})dt = \int_G f(t)dt, f \in L^1(G).
$$
How to describe

\[ C^*(G) := C^*(L^1(G)) \]
Fourier transform

The Fourier transform $F(a)$ of an element $a$ of a $\mathbb{C}^*$-algebra $A$ is defined in the following way:

One chooses for every $\gamma \in \hat{A}$ a representation $(\pi_\gamma, H_\gamma)$ in the equivalence class of $\gamma$ and let $F(a)(\gamma) := \pi_\gamma(a) \in H_\gamma$ for all $\gamma \in \hat{A}$.

Then $F(a)$ is contained in the algebra of all bounded operator fields over $\hat{A}$, $l_\infty(\hat{A}) = \left\{ \phi = \left( \phi(\pi_\gamma) \in B(H_\gamma) \right)_{\gamma \in \hat{A}} \mid \|\phi\|_\infty := \sup_{\gamma \in \hat{A}} \|\phi(\pi_\gamma)\|_\text{op} < \infty \right\}$ and the mapping $F : A \to l_\infty(\hat{A}), a \mapsto \hat{a}$ is an isometric $\ast$-homomorphism.
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and the mapping

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Examples:

Using Fourier transform:

- $G$ abelian $\Rightarrow C^*(G) \simeq C_0(\hat{G})$. 
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- **G** compact \(\Rightarrow\) \(C^*(G) \simeq C_0(\hat{G}).\)

Here

\[ C_0(\hat{G}) = \{ \varphi = (\varphi(\pi) \in B(\mathcal{H}_\pi))_{\pi \in \hat{G}} \mid \lim_{\pi \to \infty} \|\varphi(\pi)\|_{op} = 0 \}. \]
Nilpotent and exponential Lie groups

Definition

A polynomial (resp. analytic) vector group is a real finite dimensional vector space $V$ equipped with a polynomial (resp. analytic) group multiplication $\cdot$ such that

\[ sX \cdot tX = sX + tX = (s + t)X, \]

$X \in V$, $s, t \in \mathbb{R}$.

Polynomial (resp. analytic) means for any basis $B = \{Z_1, \cdots, Z_n\}$ of $V$ there exists polynomial (resp. analytic ) functions $m_j: V \times V \to V$ such that

\[ X \cdot Y = \sum_{j=1}^{n} m_j(X, Y)Z_j, \]

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Proposition

Every polynomial resp. analytic vector group is nilpotent (resp. analytic) and every connected simply connected nilpotent (resp. analytic) Lie group is isomorphic to some polynomial (resp. analytic) vector group.
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The Lie algebra of the polynomial vector group \((V, \cdot)\) can be identified with \(V\) and then

\[
X \cdot Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] + \ldots
\]
Let $\lambda_i \in \mathbb{R}^* + i\mathbb{R}$, $i = 1, \cdots, n$ and let

$T_n(\lambda_1, \cdots, \lambda_n) := 
\begin{cases}
\begin{pmatrix}
e^{t_1 \lambda_1} & \alpha_{1,2} & \cdots & \cdots & \alpha_{1,n} \\
0 & e^{t_2 \lambda_2} & \alpha_{2,3} & \cdots & \alpha_{2,n} \\
\vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & e^{t_n \lambda_n}
\end{pmatrix},
\end{cases}
\begin{cases}
t_i \in \mathbb{R}, \quad \alpha_{i,j} \in \mathbb{C}
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\right).

Theorem
Every exponential Lie group is isomorphic to a closed connected subgroup of some $T_n(\lambda_1, \cdots, \lambda_n)$. 
Let $\lambda_i \in \mathbb{R}^* + i\mathbb{R}$, $i = 1, \cdots, n$ and let

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  \vdots & 0 & \ddots & \ddots & \vdots \\
  \vdots & \vdots & \ddots & \ddots & \alpha_{n-1,n} \\
  0 & 0 & \cdots & 0 & e^{t_n \lambda_n}
\end{pmatrix}, \, t_i \in \mathbb{R}, \, \alpha_{i,j} \in \mathbb{C} \right\}.$$  

**Theorem**

*Every exponential Lie group is isomorphic to a closed connected subgroup of some $T_n(\lambda_1, \cdots, \lambda_n)$.***
Kirillov theory

Let $G = \exp(g)$ be an exponential Lie group. Let $g^*$ be the algebraic dual of $g$. The group $G$ acts by conjugation on $sts \mapsto s^{-1} t s$, $s, t \in G$, $g \in G \Rightarrow \text{Ad}(g)(X) = \frac{d}{dt}\exp(tX)g^{-1}$, $X \in g$, $g \in G \Rightarrow \text{Ad}^*(g) \ell = \text{Ad}(g^{-1}) \text{tr}_\ell$, $\ell \in g^*$.

$\langle \text{Ad}^*(g) \ell, X \rangle = \langle \ell, \text{Ad}(g^{-1})X \rangle$, $\ell \in g^*$, $X \in g$.

For any $\ell \in g^*$ let $G(\ell) = \{ g \in G ; \text{Ad}^*(g)\ell = \ell \} = \exp(\{ S \in g ; \langle \ell, [S, g] \rangle = \{0\} \}) = \exp(g(\ell))$. 
Kirillov theory

Let $G = \exp(\mathfrak{g})$ be an exponential Lie group. Let $\mathfrak{g}^*$ be the algebraic dual of $\mathfrak{g}$. The group $G$ acts by conjugation on $G$

$$t \mapsto sts^{-1}, s, t \in G,$$

$\Rightarrow Ad (g)X = \frac{d}{dt}g \exp(tX)g^{-1}, X \in \mathfrak{g}, g \in G$

$\Rightarrow Ad^*(g)\ell = Ad (g^{-1})^{tr} \ell$, i.e.

$$\langle Ad^*(g)\ell, X \rangle = \langle \ell, Ad (g^{-1})X \rangle, \ell \in \mathfrak{g}^*, X \in \mathfrak{g}.$$
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For any $\ell \in \mathfrak{g}^*$ let

$$G(\ell) = \{g \in G; Ad^*(g)\ell = \ell\}$$

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Let

$$\Omega_\ell := \{ \text{Ad}^*(g)\ell; g \in G \} \simeq G/G(\ell)$$

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the coadjoint orbit of $\ell$. Then $\Omega_\ell$ is a locally closed subset and a smooth submanifold of $\mathfrak{g}^*$ of dimension $\dim(\mathfrak{g}/\mathfrak{g}(\ell))$. 
Let

\[ \Omega_\ell := \{ \text{Ad}^*(g)\ell; g \in G \} \simeq G/G(\ell) \]

the coadjoint orbit of \( \ell \).
Then \( \Omega_\ell \) is a locally closed subset and a smooth submanifold of \( g^* \)
of dimension \( \dim(g/g(\ell)) \).
If \( G \) is nilpotent then \( \Omega_\ell \) is a Zarisky closed subset of \( g^* \).
Kirillov theory

A polarization at $\mathcal{L}$ is a subalgebra $p$ of $g$ such that

$$\langle \mathcal{L}, [p, p] \rangle = \{0\},$$

$$\dim(p) = \frac{\dim(g/g(\mathcal{L}))}{2} + \dim(g(\mathcal{L})).$$

Let $P = \exp(p)$ and

$$\chi_{\mathcal{L}}(\exp(X)) = e^{-i\langle \mathcal{L}, X \rangle}, X \in p.$$

Then $\chi_{\mathcal{L}}$ is a unitary character of the group $P$. 
Kirillov theory

A polarization at $\ell$ is a subalgebra $p$ of $g$ such that

$$\langle \ell, [p, p] \rangle = \{0\},$$

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Let $P = \exp(p)$ and

$$\chi_\ell(\exp(X)) = e^{-i \langle \ell, X \rangle}, X \in p.$$  

Then $\chi_\ell$ is a unitary character of the group $P$. We say that $p$ is a Pukanszky (Puk) representation at $\ell$ if $\text{Ad}^*(P)\ell = \ell + p^\perp$. 
Kirillov theory

So we can form the induced representation

$$\pi_{\ell, p} = \text{ind}^G_P \chi_{\ell}$$

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which acts by left translation on the space

$$L^{2}(G/P, \chi_{\ell})$$

:= \{ \xi : G \to \mathbb{C}; \xi(gp) = \chi_{\ell}(p^{-1})(\frac{\Delta_{P}(p)}{\Delta_{G}(p)})^{1/2}\xi(g), g \in G, p \in P;$$

\[\xi \text{ measurable }, \|\xi\|_{2}^{2} := \int_{G/P} |\xi(g)|^{2} d\hat{g} < \infty\}

\[\simeq L^{2}(\mathbb{R}^{m}), (m = \dim(g/p)).\]
Kirillov theory

Theorem

(Kirillov) $(G$ nilpotent) For every $\pi \in \hat{G}$ and every $c \in C^*(G)$ the operator $\pi(c)$ is compact and for every $f \in S(G)$ the operator $\pi(f)$ is trace class.
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Theorem
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Theorem
(Kirillov, Bernat, Pukanszky, Vergne) Let $G = \exp(g)$ be an exponential Lie group.

- For every $\ell \in g^*$ there exists a Puk-polarization at $\ell$.
- For every $\ell \in g^*$ and for every Puk-polarization $p$ at $\ell$ the representation $\pi_{\ell,p}$ is irreducible.
- For $\ell, \ell' \in g^*$ for any Puk-polarization $p$ at $\ell$ and $p'$ at $\ell'$ we have that

$$\pi_{\ell,p} \simeq \pi_{\ell',p'} \iff \Omega_\ell = \Omega_{\ell'}.$$

- every $\pi \in \hat{G}$ is equivalent to some $\pi_{\ell,p}$. 
Kirillov theory

Hence the mapping

$$\mathcal{K}: g^*/G \to \hat{G}; \Omega_\ell \mapsto [\pi_{\ell,p}] = [\pi_\ell]$$

is a bijection.
Kirillov theory

Hence the mapping

\[ \mathcal{K} : \mathfrak{g}^* / G \rightarrow \hat{G}; \Omega_\ell \mapsto [\pi_\ell, p] = [\pi_\ell] \]

is a bijection.

**Theorem**

*(Kirillov-Brown, Fujiwara, Leptin-L.)* The mapping \( \mathcal{K} \) is a homeomorphism.
An example: The group $G_{5,6}$
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The Lie algebra $\mathfrak{g}_{5,6}$ is spanned by the basis $\mathcal{B} = \{A, B, C, U, V\}$ equipped with the Lie brackets


The group $G_{5,6} = \exp(\mathfrak{g}_{5,6})$ can be realized as $\mathbb{R}^5$ with the multiplication

$$(a, b, c, u, v) \cdot (a', b', c', u', v') = (a + a', b + b', c + c' - a'b, u + u' - a'c + \frac{a'^2b}{2},$$

$$v + v' - a'u + \frac{bc'}{2} - \frac{b'c}{2} + \frac{a'bb'}{2} - \frac{a'^2c}{2} - \frac{a'^3b}{6}).$$ (1)
We use the Euclidean scalar product on $\mathfrak{g}_{5,6}$ to identify $\mathfrak{g}^*_{5,6}$ with $\mathfrak{g}_{5,6} = \mathbb{R}^5$ and we obtain the following expression for $\text{Ad}^*(a, b, c, u, v)$:
We use the Euclidean scalar product on \( g_{5,6} \) to identify \( g_{5,6}^* \) with \( g_{5,6} = \mathbb{R}^5 \) and we we obtain the following expression for \( \text{Ad}^*(a, b, c, u, v) \):

\[
\text{Ad}^*((a, b, c, u, v))(\alpha, \beta, \rho, \mu, \nu) = (\alpha + \rho b + \mu c - \mu \frac{ab}{2} + \nu u - \nu \frac{b^2}{2} - \nu \frac{ac}{2} + \nu \frac{a^2 b}{6},
\beta - \rho a + \mu \frac{a^2}{2} + \nu c + \nu \frac{ab}{2} - \nu \frac{a^3}{6},
\rho - \mu a - \nu b + \nu \frac{a^2}{2}, \mu - \nu a, \nu).
\]
The co-adjoint orbits:

3) The generic orbits: Let \(\nu \neq 0\). The orbit \(O_\nu\) of the element \(\ell_\nu = (0,0,0,0,\nu)\) is given by:
\[
O_\nu = \{ (a, b, c, u, \nu), a, b, c, u \in \mathbb{R} \}.
\]
We denote by \(\Gamma_{5,6,3}\) the orbit space of this layer and we parametrize it by
\[
\Gamma_{5,6,3} := \{ O_\nu \equiv \nu, \nu \in \mathbb{R}^* \}.
\]

2) Let for \((\beta, \mu) \in \mathbb{R} \times \mathbb{R}^*\) \(\ell_{\beta,\mu} := (0, \beta, 0, \mu, 0)\). Then
\[
O_{\beta,\mu} = O_{\ell_{\beta,\mu}} = \{ (a, \beta + u^2, u, \mu, 0), u, a \in \mathbb{R} \}.
\]
Let \(\Gamma_{5,6,2}\) be:
\[
\{ O_{\beta,\mu} \mid (\beta, \mu) \in \mathbb{R} \times \mathbb{R}^* \}.
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We denote by \( \Gamma_{5,6} \) the orbit space of this layer and we parametrize it by

\[
\Gamma_{3,6} := \{O_\nu \equiv \nu, \nu \in \mathbb{R}^*\}. \tag{2}
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The co-adjoint orbits:

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$$\mathcal{O}_\nu = \{(a, b, c, u, \nu), \ a, b, c, u \in \mathbb{R}\}.$$  

We denote by $\Gamma_{3,6}^{5,6}$ the orbit space of this layer and we parametrize it by

$$\Gamma_{3,6}^{5,6} := \{\mathcal{O}_\nu \equiv \nu, \nu \in \mathbb{R}^*\}. \quad (2)$$

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Then

$$\mathcal{O}_{\beta,\mu} = \mathcal{O}_{\ell_{\beta,\mu}} = \{(a, \beta + \frac{u^2}{2\nu}, u, \mu, 0)| u, a \in \mathbb{R}\}.$$
The co-adjoint orbits:

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2) Let for \( (\beta, \mu) \in \mathbb{R} \times \mathbb{R}^* \)

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Then

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\mathcal{O}_{\beta,\mu} = \mathcal{O}_{\ell_{\beta,\mu}} = \{ (a, \beta + \frac{u^2}{2\nu}, u, \mu, 0) | u, a \in \mathbb{R} \}.
\]

Let

\[
\Gamma_{2,6}^{5,6} := \{ \mathcal{O}_{\beta,\mu} | (\beta, \mu) \in \mathbb{R} \times \mathbb{R}^* \}
\]
1) Let for $\sigma \in \mathbb{R}^*$

$$l_\nu := (0, 0, \sigma, 0, 0).$$

Then

$$O_\sigma = O_{l_\sigma} = \{(a, b, \sigma, 0, 0) | a, b \in \mathbb{R}\}.$$ 

Let

$$\Gamma_{1}^{5,6} := \{O_\sigma | \sigma \in \mathbb{R}^*\}$$

0) Let

$$\Gamma_{0}^{5,6} := \{(\alpha, \beta, 0, 0, 0) | \alpha, \beta \in \mathbb{R}\}.$$
Theorem

Let $\overline{O} = (O_{\nu_k})_k \subset \Gamma^{5,6}_3$ be a sequence, such that $\lim_{k \to \infty} \nu_k = 0$. If the sequence $\overline{O}$ is properly converging then

$$L(\overline{O}) = \Gamma^{5,6}_2 \cup \Gamma^{5,6}_1 \cup \Gamma^{5,6}_0 = \mathcal{V}^\perp.$$
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L(\overline{O}) = \Gamma_{2,6} \cup \Gamma_{1,6} \cup \Gamma_{0,6} = \mathcal{V}^\perp.
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Let \( \overline{O} = (\mathcal{O}_{\beta_k,\mu_k})_k \subset \Gamma_{2,6} \) be a properly converging sequence with \( \lim_{k \to \infty} \mu_k = 0 \).
Theorem

Let $\mathcal{O} = (O_{\nu_k})_k \subset \Gamma^{5,6}_3$ be a sequence, such that $\lim_{k \to \infty} \nu_k = 0$. If the sequence $\mathcal{O}$ is properly converging then

$$L(\mathcal{O}) = \Gamma^{5,6}_2 \cup \Gamma^{5,6}_1 \cup \Gamma^{5,6}_0 = V^\perp.$$

Let $\mathcal{O} = (O_{\beta_k, \mu_k})_k \subset \Gamma^{5,6}_2$ be a properly converging sequence with $\lim_{k \to \infty} \mu_k = 0$.

Then

$$\text{sign}(\mu_k)$$ is constant and $\lim_{k \to \infty} \beta_k \mu_k = \kappa$ exists.
If $\kappa \neq 0$, then $\kappa < 0$ and

$$L(O) = O_{\sqrt{-\kappa}} \cup O_{-\sqrt{-\kappa}}.$$
If $\kappa \neq 0$, then $\kappa < 0$ and

$$L(\mathcal{O}) = \mathcal{O}_{\sqrt{-\kappa}} \cup \mathcal{O}_{-\sqrt{-\kappa}}.$$ 

If $\kappa = 0$ and $\nu_k > 0$, $k \in \mathbb{N}$, then $\beta_\infty = \lim_{k \to \infty} \beta_k$ exists in $[-\infty, \infty]$ and $L(\mathcal{O}) = \mathbb{R}A^* + [\beta_\infty, \infty][B^*].$
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If $\kappa \neq 0$, then $\kappa < 0$ and

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If $\kappa = 0$ and $\nu_k > 0$, $k \in \mathbb{N}$, then $\beta_\infty = \lim_{k \to \infty} \beta_k$ exists in $[-\infty, \infty]$ and $L(\mathcal{O}) = \mathbb{R}A^* + [\beta_\infty, \infty][B^*$. 

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Let $\mathcal{O} = (\mathcal{O}_{\sigma_k})_k \subset \Gamma^{5,6}_{1}$ be a properly converging sequence with $\lim_{k \to \infty} \sigma_k = 0$. Then $L(\mathcal{O}) = \mathbb{R}A^* + \mathbb{R}B^*$. 
An example: The $C^*$-algebra of the Heisenberg group

Let $H_n = \mathbb{R}^{2n+1}$ on which the multiplication is given by

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + \frac{1}{2}(x \cdot y' - x' \cdot y)), $$

where $x \cdot y = x_1 y_1 + \cdots + x_n y_n$ denotes the Euclidean scalar product on $\mathbb{R}^n$. 
An example: The $C^\ast$-algebra of the Heisenberg group

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The centre of $H_n$ is the subgroup $\mathcal{Z} := \{0_n\} \times \{0_n\} \times \mathbb{R}$ and the commutator subgroup $[H_n, H_n]$ of $H_n$ is given by $[H_n, H_n] = \mathcal{Z}$. 
The Lie algebra $\mathfrak{g}$ of $H_n$ has the basis

$$\mathcal{B} := \{ X_j, Y_j, j = 1 \cdots, n, Z = (0_n, 0_n, 1) \},$$

where $X_j = (e_j, 0_n, 0), Y_j = (0_n, e_j, 0), j = 1, \cdots, n$ and $e_j$ is the $j$'th canonical basis vector of $\mathbb{R}^n$, with the non trivial brackets

$$[X_i, Y_j] = \delta_{i,j} Z.$$
The unitary dual of $H_n$.

The infinite dimensional irreducible representations:
For every $\lambda \in \mathbb{R}^*$, there exists a unitary representation $\pi_\lambda$ of $H_n$ on the Hilbert space $L^2(\mathbb{R}^n)$, which is given by the formula

$$
\pi_\lambda(x, y, t) \xi(s) := e^{-2\pi i \lambda t} - 2\pi i \lambda x \cdot y + 2\pi i \lambda s \cdot y \xi(s - x),
$$

$s \in \mathbb{R}^n, \xi \in L^2(\mathbb{R}^n), (x, y, t) \in H_n$.

It is easily seen that $\pi_\lambda$ is in fact irreducible and that $\pi_\lambda$ is equivalent to $\pi_\nu$ if and only if $\lambda = \nu$.

The representation $\pi_\lambda$ is equivalent to the induced representation $\tau_\lambda := \text{ind}_{H_n}^{P} \chi_\lambda$, where $P = \{0^n\} \times \mathbb{R}^n \times \mathbb{R}$ is a polarization at the linear functional $\ell_\lambda((x, y, t)) := \lambda t, (x, y, t) \in g$ and where $\chi_\lambda$ is the character of $P$ defined by $\chi_\lambda(0^n, y, t) = e^{-2\pi i \lambda t}$.

The theorem of Stone-Von Neumann (1929) tells us that every infinite dimensional unitary representation of $H_n$ is equivalent to one of the $\pi_\lambda$'s.
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The finite dimensional irreducible representations

Since $H_n$ is nilpotent, every irreducible finite dimensional representation of $H_n$ is one-dimensional, by Lie’s theorem. Any one-dimensional representation is a unitary character $\chi_{a,b}$, $(a, b) \in \mathbb{R}^n \times \mathbb{R}^n$, of $H_n$, which is given by

$$\chi_{a,b}(x, y, t) = e^{-2\pi i (a \cdot x + b \cdot y)}, (x, y, t) \in H_n.$$
The Fourier transform

Definition
Define for $c \in C^*(H_n)$ the Fourier transform $F(c)$ of $c$ by

$$\hat{c}(\lambda) = \hat{c}(\pi_\lambda) = F(c)(\lambda) := \pi_\lambda(c) \in B(L^2(\mathbb{R}^n)), \lambda \in \mathbb{R}^*$$

and

$$\hat{c}(u) = F(c)(u) := \chi_u(c), \; u \in \mathbb{R}^{2n}.$$
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Define for $c \in C^*(H_n)$ the function

$$\hat{c}(0)(u) := \hat{c}(u), \; u \in \mathbb{R}^{2n}. $$

Then the mapping

$$C^*(H_n) \rightarrow C_0(\mathbb{R}^{2n}); \; c \mapsto \hat{c}(0),$$

is a surjective homomorphism.
The topology of $C^* (H_n)$

As for the topology of the dual space, it is well known that $[\pi_\lambda]$ tends to $[\pi_\nu]$ in $\hat{H}_n$ if and only if $\lambda$ tends to $\nu$ in $\mathbb{R}^*$, where $[\pi]$ denotes the unitary equivalence class of the unitary representation $\pi$. Furthermore, if $\lambda$ tends to 0, then the representations $\pi_\lambda$ converge in the dual space topology to all the characters $\chi_{a,b}$, $a, b \in \mathbb{R}^n$. 
Choose a Schwartz-function $\eta$ in $S(\mathbb{R}^n)$ with $L^2$-norm equal to 1. For $u = (a, b)$ in $\mathbb{R}^n \times \mathbb{R}^n$, $\lambda \in \mathbb{R}^*$, we define the function $\eta(\lambda, a, b)$ by

$$
\eta(\lambda, a, b)(s) := |\lambda|^{n/4} e^{2\pi i a \cdot s} \eta(|\lambda|^{1/2} (s + \frac{b}{\lambda})), \ s \in \mathbb{R}^n, \quad (3)
$$

and let $\eta_\lambda(s) = |\lambda|^{n/4} \eta(|\lambda|^{1/2} s), \ s \in \mathbb{R}^n.$
\[ \lim_{\lambda \to 0} \pi \lambda \]

Let us compute

\[
\langle \pi \lambda(x, y, t) \eta(\lambda, u), \eta(\lambda, u) \rangle
= \int_{\mathbb{R}^n} e^{-2\pi i \lambda t - 2\pi i (\lambda/2) x \cdot y} e^{2\pi i \lambda s \cdot y} \eta(\lambda, u)(s - x) \frac{\eta(\lambda, u)(s) ds}{\eta(\lambda, u)(s)}
= e^{-2\pi i \lambda t - 2\pi i \lambda x \cdot y} e^{-2\pi i a \cdot x - 2\pi i b \cdot y}
\int_{\mathbb{R}^n} e^{2\pi i (\text{sign}\lambda) |\lambda|^{1/2} s \cdot y} \eta(s - |\lambda|^{1/2} x) \overline{\eta(s)} ds
\to e^{-2\pi i a \cdot x - 2\pi i b \cdot y} \int_{\mathbb{R}^n} \eta(s) \overline{\eta(s)} ds = e^{-2\pi i a \cdot x - 2\pi i b \cdot y}.\]
Let us compute

\[
\langle \pi_{\lambda}(x, y, t)\eta(\lambda, u), \eta(\lambda, u) \rangle = \int_{\mathbb{R}^n} e^{-2\pi i \lambda t - 2\pi i (\lambda/2)x \cdot y} e^{2\pi i \lambda s \cdot y} \eta(\lambda, u)(s - x) \frac{\eta(\lambda, u)(s)}{\eta(\lambda, u)(s)} ds
\]

\[
= e^{-2\pi i \lambda t - 2\pi i \frac{\lambda}{2} x \cdot y} e^{-2\pi i a \cdot x - 2\pi i b \cdot y} \int_{\mathbb{R}^n} e^{2\pi i (\text{sign} \lambda)|\lambda|^{1/2} s \cdot y} \eta(s - |\lambda|^{1/2} x)\overline{\eta(s)} ds
\]

\[
\rightarrow e^{-2\pi i a \cdot x - 2\pi i b \cdot y} \int_{\mathbb{R}^n} \eta(s)\overline{\eta(s)} ds = e^{-2\pi i a \cdot x - 2\pi i b \cdot y}.
\]

It follows also that the convergence of the coefficients to the characters \(\chi_{a, b}\) is uniform in \(u\) and uniform on compacta.
Properties of the Fourier transform

Let us compute for $f \in L^1(H_n)$ the operator $\pi_\lambda(f)$. We have for $\xi \in L^2(\mathbb{R}^n)$ and $s \in \mathbb{R}^n$ that

$$\pi_\lambda(f)\xi(s) = \int_{\mathbb{R}^n} \hat{f}^{2,3}(s - x, -\frac{\lambda}{2}(s + x), \lambda)\xi(x)dx.$$
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Here

$$\hat{f}^{2,3}(s, u, \lambda) = \int_{\mathbb{R}^n \times \mathbb{R}} f(s, y, t)e^{-2\pi i(y \cdot u + \lambda t)}dydt, \ (s, u, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$$

denotes the partial Fourier transform of $f$ in the variables $y$ and $t$. 

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denotes the partial Fourier transform of $f$ in the variables $y$ and $t$. Hence $\pi_\lambda(f)$ is a kernel operator with kernel

$$f_\lambda(s, x) := \hat{f}^{2,3}(s-x, -\frac{\lambda}{2}(s+x), \lambda), s, x \in \mathbb{R}^n.$$ (4)
Properties of the Fourier transform

If we take now a Schwartz-functions \( f \in S(H_n) \), then the operator \( \pi_\lambda(f) \) is Hilbert-Schmidt and its Hilbert-Schmidt norm \( \|\pi_\lambda(f)\|_{\text{H.S.}} \) is given by

\[
\|\pi_\lambda(f)\|_{\text{H.S.}}^2 = \int_{\mathbb{R}^2} |f_\lambda(s, x)|^2 \, dx \, ds = \int_{\mathbb{R}^2} |\hat{f}^{2,3}(s, \lambda x, \lambda)|^2 \, ds \, dx < \infty.
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Proposition

For any $c \in C^*(H_n)$ and $\lambda \in \mathbb{R}^*$, the operator $\pi_\lambda(c)$ is compact, the mapping $\mathbb{R}^* \to B(L^2(\mathbb{R}^n)) : \lambda \mapsto \pi_\lambda(c)$ is norm continuous and tending to 0 for $\lambda$ going to infinity.
The condition in 0 and the mappings $\sigma_\lambda$
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Recall that for $\eta \in S(\mathbb{R}^n)$, $\|\eta\|_2 = 1$, $u = (a, b) \in \mathbb{R}^n \times \mathbb{R}^n$, $\lambda \in \mathbb{R}^*$, we have defined

$$\eta(\lambda, a, b)(s) := |\lambda|^{n/4} e^{2\pi i a \cdot s} \eta(|\lambda|^{1/2}(s + \frac{b}{\lambda})) \ s \in \mathbb{R}^n$$
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$$

Simple calculation shows that

$$
\pi_\lambda(f) = \frac{1}{|\lambda|^n} \int_{\mathbb{R}^{2n}} \pi_\lambda(f) \circ P_{\eta(\lambda, u)} du
$$

for any $f \in S(H_n)$, where $P_{\eta(\lambda, u)}$ is the orthogonal projection onto the one dimensional subspace $\mathbb{C}\eta(\lambda, u)$. 
The condition in $0$ and the mappings $\sigma_\lambda$

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**Definition**

Define for $\lambda \in \mathbb{R}^*$ and $h \in C_0(\mathbb{R}^{2n})$ the linear operator

$$\sigma_\lambda(h) := \frac{1}{|\lambda|^n} \int_{\mathbb{R}^{2n}} h(u) P_{\eta(\lambda, u)} du.$$
Theorem
(L., Turowska 2008)

1. $\sigma_\lambda(h)$ is compact and $\|\sigma_\lambda(h)\|_{op} \leq \|h\|_{\infty}$.

2. $\sigma_\lambda(h^*)(\lambda) = \sigma_\lambda(h)^*$, $h \in C_0(\mathbb{R}^{2n})$.

3. $\|\pi_\lambda(a) - \sigma_\lambda(F(a)(0))\|_{op} \to 0$ as $\lambda \to 0$ for each $a \in C^*(H_n)$. 
Definition
Let $D_\nu^*(H_n)$ be the space consisting of all the fields $(F(\lambda))_{\lambda \in \mathbb{R}}$, such that

1. $F(\lambda)$ is a compact operator on $L^2(\mathbb{R}^n)$ for every $\lambda \in \mathbb{R}^*$,
2. $F(0) \in C^*(\mathbb{R}^2),$
3. the mapping $\mathbb{R}^* \to B(L^2(\mathbb{R}^n)) : \lambda \mapsto F(\lambda)$ is norm continuous,
4. $\lim_{\lambda \to \infty} \|F(\lambda)\|_{\text{op}} = 0,$
5. $\lim_{\lambda \to 0} \|F(\lambda) - \sigma_\lambda(F(0))\|_{\text{op}} = 0.$
We have the following characterisation of $C^*(H_n)$.

**Theorem**

(L., Turowska) The space $D^*_\nu(H_n)$ is a $C^*$-algebra which is isomorphic to $C^*(H_n)$ under the Fourier transform.
Norm controlled dual limits

**Theorem**

(I. Beltita, B. Beltita, J. L.) Let $A$ be a separable liminary $C^*$-algebra, let $\pi = \{(\pi_k, \mathcal{H}_k)\}_{k \in \mathbb{N}}$ be a properly converging sequence in $\hat{A}$ with limit set $L$. There exists a uniformly bounded sequence of bounded linear mappings $\tilde{\sigma}_k: F(A) \mid L \to B(H_k)$ such that

$$\lim_{k \to \infty} \| \tilde{\sigma}_k(\hat{a}\mid L) - \pi_k(a) \|_{op} = 0,$$

for all $a \in A$. 
Norm controlled dual limits

**Theorem**
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$$\tilde{\sigma}_k : \mathcal{F}(\mathcal{A})|_L \to \mathcal{B}(\mathcal{H}_k)$$

such that

$$\lim_{k \to \infty} \| \tilde{\sigma}_k(\hat{a}|_L) - \pi_k(a) \|_{op} = 0, \ a \in \mathcal{A}.$$
The problem is how to find a precise expression for the mappings $\tilde{\sigma}_k$, whenever the algebra $\mathcal{A}$ and the representations $\pi_k$ are concretely given.
Definition i) Let $S$ be a topological space. We say that $S$ is *locally compact of step* $\leq N$ if there exists a finite increasing family $\emptyset \neq S_0 \subset \cdots \subset S_N = S$ of closed subsets of $S$, such that the subsets $\Gamma_0 = S_0$ and $\Gamma_i := S_i \setminus S_{i-1}$, $i = N, \ldots, 1$, are locally compact and Hausdorff in their relative topologies.
ii) Let $S$ be locally compact of step $\leq d$, and let $\{\mathcal{H}_i\}_{i=1,\ldots,d}$ be Hilbert spaces. For a closed subset $M \subset S$, denote by $CB(M)$ the unital $C^*$-algebra of all uniformly bounded operator fields $(\psi(\gamma) \in \mathcal{B}(\mathcal{H}_i))_{\gamma \in S \cap \Gamma_i, i=1,\ldots,d}$, which are operator norm continuous on the subsets $\Gamma_i \cap M$ for every $i \in \{1, \ldots, d\}$ with $\Gamma_i \cap M \neq \emptyset$. 
$C^*$-algebras with norm controlled dual limits

ii) Let $S$ be locally compact of step $\leq d$, and let \{${\mathcal H}_i$\}$_{i=1,\ldots,d}$ be Hilbert spaces. For a closed subset $M \subset S$, denote by $CB(M)$ the unital $C^*$-algebra of all uniformly bounded operator fields

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$$\|\varphi\|_M = \sup \left\{ \|\varphi(\gamma)\|_{B(\mathcal H_i)} \mid M \cap \Gamma_i \neq \emptyset, \gamma \in M \cap \Gamma_i \right\}.$$
Spaces of operator fields with norm controlled dual limits

Definition
Let $\emptyset = S_{-1} \subset S_0 \subset \cdots \subset S_d = S$ be a locally compact topological space of step $\leq d$. Choose for every $i = 1, \ldots, d$ a Hilbert space $\mathcal{H}_i$ and assume that $\mathcal{H}_0 = \mathbb{C}$. Let $B^*(S)$ be the set of all operator fields $\varphi$ defined over $S$ such that

- $\varphi(\gamma) \in \mathcal{K}(\mathcal{H}_i)$ for every $\gamma \in \Gamma_i = S_{i-1} \setminus S_i$, $i = 1, \ldots, d$.
- The field $\varphi$ is uniformly bounded, that is, we have that
  $$\|\varphi\| = \sup \{ \|\varphi(\gamma)\|_{B(\mathcal{H}_i)} \mid \gamma \in \Gamma_i, i = 1, \ldots, d \} < \infty.$$
- The mappings $\gamma \rightarrow \varphi(\gamma)$ are norm continuous on the difference sets $\Gamma_i$. 
We have for any sequence \((\gamma_k)_{k \in \mathbb{N}} \subset S\) going to infinity, that

\[
\lim_{k \to \infty} \| \varphi(\gamma_k) \|_{\text{op}} = 0.
\]

For any \(i = 1, \ldots, d + 1\), and for any converging sequence contained in \(\Gamma_i = S_i \setminus S_{i-1}\) with limit set outside \(\Gamma_i\), there exists a properly converging sub-sequence \(\gamma = (\gamma_k)_{k \in \mathbb{N}}\), a constant \(C > 0\) and for every \(k \in \mathbb{N}\) an involutive linear mapping \(\tilde{\sigma}_{\gamma,k} : CB(S_i) \to \mathcal{B}(\mathcal{H}_i)\), which is bounded by \(C \| \cdot \|_{S_i}\), such that

\[
\lim_{k \to \infty} \| \varphi(\gamma_k) - \tilde{\sigma}_{\gamma,k}(\varphi|_{S_i}) \|_{\mathcal{B}(\mathcal{H}_i)} = 0.
\]
Theorem
Let \( S \) be a locally compact topological space of step \( \leq d \). Then the set \( B^*(S) \) of the preceding frame is a closed involutive subspace of \( \ell^\infty(S) \). Furthermore \( B^*(S) \) is a \( C^* \)-subalgebra of \( \ell^\infty(S) \) with spectrum \( S \) if and only if all the mappings \( \tilde{\sigma}_{\tilde{\gamma},k} \) are almost homomorphisms, i.e.,

\[
\lim_{k \to \infty} \| \tilde{\sigma}_{\tilde{\gamma},k}(\varphi \cdot \psi) - \tilde{\sigma}_{\tilde{\gamma},k}(\varphi) \cdot \tilde{\sigma}_{\tilde{\gamma},k}(\psi) \|_{\mathcal{B}(\mathcal{H}_i)} = 0, \quad \varphi, \psi \in B^*(S),
\]

and the restrictions \( B^*(S)|_{S_{i-1}} \) contain the spaces \( C_0(\Gamma_i, \mathcal{H}_i) \), \( i = 1, \ldots, d + 1 \).
Definition
Let us denote by $\tilde{\sigma}$ the collection of all these mappings $\tilde{\sigma} \gamma$ and by

$$C_0(S, \tilde{\sigma})$$

the $C^*$-algebra of operator fields defined over $S$ satisfying the conditions of the preceding theorems.
Definition
Let $A$ be a separable liminary $C^*$-algebra. We say that the $C^*$-algebra $A$ has norm controlled dual limits if $A$ is isomorphic to some $C_0(S, \tilde{\sigma})$-$C^*$-algebra.
Theorem
(I. Beltita, D. Beltita, L-) Let $G$ be a connected simply connected nilpotent Lie group. Then its spectrum $\hat{G}$ is locally compact of step $d$ for some $d \geq 0$. 
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The $C^*$-algebra $C^*(G)$ has norm controlled dual limits, i.e. is isomorphic to $C^*(\hat{G}, \tilde{\sigma})$ for some family $\tilde{\sigma}$.
An exponential example: The groups $G_{n,\mu}$.

Write

$$\mathfrak{h}_n := V_n \oplus \mathbb{R}.$$
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Write

$$\mathfrak{h}_n := \mathfrak{v}_n \oplus \mathbb{R}.$$ 

Let

$$\mathfrak{g}_{n,\mu} := \mathbb{R} \times \mathfrak{h}_n \text{ and } A = (1, 0_{\mathfrak{v}_n}, 0), \ Z = (0, 0_{\mathfrak{v}_n}, 1) \in \mathfrak{g}_{n,\mu}.$$
An exponential example: The groups $G_{n,\mu}$.

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For

$$\mu := \{\lambda_1, \lambda'_1, \ldots, \lambda_n, \lambda'_n\} \subset \mathbb{R}$$

with $\lambda_i + \lambda'_i = 2$ for all $i = 1, \cdots, n$, we define the brackets

$$[A, X_i] = \lambda_i X_i, \quad [A, Y_i] = \lambda'_i Y_i, \quad [A, Z] = 2Z \quad \text{for all } i = 1, \cdots, n,$$

and

$$[X_i, Y_j] = \delta_{i,j} Z \quad \text{for } i, j = 1, \cdots, n.$$
We then obtain a structure of an exponential solvable Lie algebra on $g_{n,\mu}$, and its subalgebra $h_n$ is the Heisenberg Lie algebra. Define the diagonal operator $l_\mu : V_n \to V_n$ by

$$l_\mu(v) := \sum_{i} \lambda_i v_i X_i + \lambda'_i v'_i Y_i \quad \text{for} \quad v = \sum_{i=1}^{n} v_i X_i + \sum_{i=1}^{n} v'_i Y_i \in V_n.$$
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Define the diagonal operator $l_{\mu} : V_n \to V_n$ by

$$l_{\mu}(v) := \sum_{i} \lambda_i v_i X_i + \lambda_i' v_i' Y_i \quad \text{for} \quad v = \sum_{i=1}^{n} v_i X_i + \sum_{i=1}^{n} v_i' Y_i \in V_n.$$  

For $v = \sum_{i=1}^{n} v_i X_i + v_i' Y_i \in V_n$ and $a \in \mathbb{R}$, we write

$$a \cdot v := \sum_{i=1}^{n} e^{a\lambda_i} v_i X_i + e^{a\lambda_i'} v_i' Y_i.$$
We then obtain a structure of an exponential solvable Lie algebra on $\mathfrak{g}_{n,\mu}$, and its subalgebra $\mathfrak{h}_n$ is the Heisenberg Lie algebra. Define the diagonal operator $l_\mu : V_n \to V_n$ by

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$$a \cdot v := \sum_{i=1}^{n} e^{a \lambda_i} v_i X_i + e^{a \lambda'_i} v'_i Y_i.$$ 

The corresponding simply connected Lie group $G_{n,\mu}$, which is exponential solvable, can be identified with the space $\mathbb{R} \times V_n \times \mathbb{R}$ equipped with the multiplication

$$(a, v, c) \cdot (a', v', c')$$

$$= (a + a', (-a') \cdot v + v', e^{-2a'} c + c' + \frac{1}{2} \omega_n((-a') \cdot v, v')).$$
Denote by $G_{V_n}$ the quotient group $G_{n,\mu}/\mathbb{Z}$ which can be identified with $\mathbb{R} \times V_n$ equipped with the multiplication

$$(s, v) \cdot (t, w) := (s + t, (-t) \cdot v + w).$$
Denote by $G_{V_n}$ the quotient group $G_{n,\mu}/\mathbb{Z}$ which can be identified with $\mathbb{R} \times V_n$ equipped with the multiplication
\[(s, v) \cdot (t, w) := (s + t, (-t) \cdot v + w) .\]

We write $V_n = V_0 \oplus V_+ \oplus V_- = V_0 \oplus V_1$, where
\[V_+ := \text{span}\{X_j, Y_k; \lambda_j > 0, \lambda'_k > 0\},\]
\[V_- := \text{span}\{X_j; \lambda_j < 0\},\]
\[V_0 := \text{span}\{X_j, Y_k; \lambda_j = 0, \lambda'_k = 0\},\]
and $V_1 := V_+ \oplus V_-$. Let
\[\mu_+ := \mu \cap \mathbb{R}^*, \mu_- := \mu \cap \mathbb{R}^*, \mu_0 := \mu \cap \{0\},\]
then we can write
\[V_+ = \sum_{\lambda \in \mu_+} V_{+,\lambda} \text{ and } V_- = \sum_{\lambda \in \mu_-} V_{-,\lambda},\]
where $V_{+,\lambda}$ and $V_{-,\lambda}$ are the respective eigenspaces of the operator $l_\mu$. 
We can also identify $g^*_{n,\mu}$ with $\mathbb{R}A^* \oplus V_n^* \oplus \mathbb{R}Z^* \simeq \mathbb{R} \times V_n \times \mathbb{R}$, and then

\[
\langle \text{Ad}^*(a, u)(a^*, v^*, \lambda^*), (0, v, z) \rangle \\
= \langle (a^*, v^*, \lambda^*), \text{Ad}((a, u)^{-1})(0, v, z) \rangle \\
= \langle (a^*, v^*, \lambda^*), (0, (-a) \cdot v, e^{-2a}z + e^{-2a}\omega_n(-(a \cdot u), v)) \rangle \\
= \langle 0, v^*, (-a) \cdot v \rangle + \lambda^* e^{-2a}z + \lambda^* e^{-2a}\omega_n(-(a \cdot u), v).
\]
We can also identify $g_{n,\mu}$ with $\mathbb{R}A^* \oplus V_n^* \oplus \mathbb{R}Z^* \simeq \mathbb{R} \times V_n \times \mathbb{R}$, and then

$$\langle \text{Ad}^* (a, u)(a^*, v^*, \lambda^*), (0, v, z) \rangle$$

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$$= \langle (a^*, v^*, \lambda^*), (0, (-a) \cdot v, e^{-2a}z + e^{-2a}\omega_n(-(a \cdot u), v)) \rangle$$

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Hence

$$\text{Ad}^* (a, u)(a^*, v^*, \lambda^*)|_{\mathfrak{h}_n} = (a^*, (-a) \cdot v^* - \lambda^* e^{-2a}(a \cdot u) \times \omega_n, \lambda^* e^{-2a}).$$
We can also identify \( g_{n,\mu}^* \) with \( \mathbb{R} A^* \oplus V_n^* \oplus \mathbb{R} Z^* \cong \mathbb{R} \times V_n \times \mathbb{R}, \) and then

\[
\langle \text{Ad}^*(a, u)(a^*, v^*, \lambda^*), (0, v, z) \rangle \\
= \langle (a^*, v^*, \lambda^*), \text{Ad} ((a, u)^{-1})(0, v, z) \rangle \\
= \langle (a^*, v^*, \lambda^*), (0, (-a) \cdot v, e^{-2a}z + e^{-2a} \omega_n(-(a \cdot u), v)) \rangle \\
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\]

Hence

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\]

Here we denote by \( u \times \omega_n \) the linear functional on \( V_n \) as

\[
u \times \omega_n(v) := \omega_n(u, v) \quad \text{for all} \quad v \in V_n.
\]
The coadjoint orbit $\Omega_\ell$ of an element $\ell = (a^*, v^*, \lambda^*) \in g^*_n, \mu$ is given by

$$\Omega_\ell = \{(a^* + v^*([A, u]) + 2z\lambda^*, (-a) \cdot v^* - \lambda^* e^{-2a}(a \cdot u) \times \omega_n, \lambda^* e^{-2a})$$

$$|a, z \in \mathbb{R}, u \in V_n\}.$$
The coadjoint orbit \( \Omega_\ell \) of an element \( \ell = (a^*, v^*, \lambda^*) \in g^*_{n,\mu} \) is given by

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\Omega_\ell = \{ (a^* + v^*([A, u]) + 2z\lambda^*, (-a)\cdot v^* - \lambda^* e^{-2a}(a \cdot u) \times \omega_n, \lambda^* e^{-2a}) 
\mid a, z \in \mathbb{R}, u \in V_n \}.
\]

Hence if \( \lambda^* \neq 0 \) then the corresponding coadjoint orbit is the subset

\[
\Omega_{\lambda^*} = \mathbb{R} \times V^*_n \times \mathbb{R}^*_+ \lambda^*,
\]

where \( V^*_n \) is the linear dual space of \( V_n \).
The coadjoint orbit $\Omega_\ell$ of an element $\ell = (a^*, v^*, \lambda^*) \in g_{n,\mu}^*$ is given by

$$\Omega_\ell = \{(a^* + v^*([A, u]) + 2z\lambda^*, (-a) \cdot v^* - \lambda^* e^{-2a}(a \cdot u) \times \omega_n, \lambda^* e^{-2a}) \mid a, z \in \mathbb{R}, u \in V_n\}.$$ 

Hence if $\lambda^* \neq 0$ then the corresponding coadjoint orbit is the subset

$$\Omega_{\lambda^*} = \mathbb{R} \times V_n^* \times \mathbb{R}^*_+ \lambda^*,$$

where $V_n^*$ is the linear dual space of $V_n$. Therefore we have two open coadjoint orbits

$$\Omega_\varepsilon := \text{Ad}^*(G_{n,\mu})_{\ell_\varepsilon} = \mathbb{R} \times V_n^* \times \mathbb{R}^*_\varepsilon \quad \text{for} \quad \varepsilon \in \{+, -\},$$

where $\ell_\varepsilon = \varepsilon Z^*$. 

(7)
The other orbits are contained in $Z^\perp$ of the form

$$\Omega_{v^*} = \mathbb{R}A^* + \mathbb{R} \cdot v^* \text{ for } v^* \in V_n^* \setminus V_0^*,$$
The other orbits are contained in $Z^\perp$ of the form

$$\Omega_{\nu^*} = \mathbb{R}A^* + \mathbb{R} \cdot \nu^* \text{ for } \nu^* \in V_n^* \setminus V_0^*,$$

or the one point orbits

$$\{a^*A^* + \nu^*\} \text{ for } a^* \in \mathbb{R}, \nu^* \in V_0^*.$$
We can decompose the linear dual space $V^*_n$ of $V_n$ into

\[
V^*_+ := \{ f \in V^*_n : f(V_+ \cup V_0) = \{0\} \},
\]

\[
V^*_- := \{ f \in V^*_n : f(V_+ \cup V_0) = \{0\} \},
\]

\[
V^*_0 := \{ f \in V^*_n : f(V_+ \cup V_-) = \{0\} \}.
\]
We can decompose the linear dual space $V_n^*$ of $V_n$ into

\[ V_+^* := \{ f \in V_n^* : f(V_- \cup V_0) = \{0\} \}, \]
\[ V_-^* := \{ f \in V_n^* : f(V_+ \cup V_0) = \{0\} \}, \]
\[ V_0^* := \{ f \in V_n^* : f(V_+ \cup V_-) = \{0\} \}. \]

Denote by $\| \cdot \|$ the norm on $V_n^*$ coming from the scalar product defined by the basis $\{X_1, \cdots, X_n, Y_1, \cdots, Y_n\}$. For $f_+ = \sum_{\lambda \in \mu_+} f_\lambda \in V_+^*$ and $f_- = \sum_{\lambda \in \mu_-} f_\lambda \in V_-^*$, let
We can decompose the linear dual space $V_n^*$ of $V_n$ into

\[ V^*_+ := \{ f \in V_n^* : f(V^- \cup V_0) = \{0\}\}, \]
\[ V^*_- := \{ f \in V_n^* : f(V^+ \cup V_0) = \{0\}\}, \]
\[ V^*_0 := \{ f \in V_n^* : f(V^+ \cup V^-) = \{0\}\}. \]

Denote by $\| \cdot \|$ the norm on $V_n^*$ coming from the scalar product defined by the basis $\{X_1, \cdots, X_n, Y_1, \cdots, Y_n\}$. For

\[ f_+ = \sum_{\lambda \in \mu_+} f_\lambda \in V^*_+ \text{ and } f_- = \sum_{\lambda \in \mu_-} f_\lambda \in V^*_-, \]

let

\[ |f_+|_\mu = |f_+| := \max_{\lambda_j \in \mu_+} \|f_j\|^{1/\lambda_j} \text{ and } |f_-|_\mu = |f_-| := \max_{\lambda_j \in \mu_-} \|f_j\|^{-1/\lambda_j}. \]
Then for $t \in \mathbb{R}$, we have the relation

$$|t \cdot f_+| = e^t|f_+| \quad \text{and} \quad |t \cdot f_-| = e^{-t}|f_-| \quad \text{for} \quad f_+ \in V^*_+, f_- \in V^*_-. \quad (8)$$
Then for $t \in \mathbb{R}$, we have the relation

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On $V_0^*$ we shall use the norm coming from the scalar product.
Then for $t \in \mathbb{R}$, we have the relation

$$|t \cdot f_+| = e^t|f_+| \quad \text{and} \quad |t \cdot f_-| = e^{-t}|f_-| \quad \text{for} \quad f_+ \in V^*_+, \ f_- \in V^*_-. \quad (8)$$

On $V^*_0$ we shall use the norm coming from the scalar product. This gives us a global gauge on $V^*_n$:

$$|(f_0, f_+, f_-)| := \max\{\|f_0\|, |f_+|, |f_-|\}.$$
We denote by $V^*_{gen}$ the open subset of $V^*_n$ consisting of all the $f = (f_0, f_+, f_-) \in V^*_0 \times V^*_+ \times V^*_-$ for which $f_+ \neq 0$ and $f_- \neq 0$. The subset $V^*_sin$ consists of all the $f = (f_0, f_+, f_-)$ for which either $f_+ \neq 0, f_- = 0$ or $f_+ = 0, f_- \neq 0$. 
We denote by $V_{gen}^*$ the open subset of $V_n^*$ consisting of all the $f = (f_0, f_+, f_-) \in V_0^* \times V_+^* \times V_-^*$ for which $f_+ \neq 0$ and $f_- \neq 0$. The subset $V_{sin}^*$ consists of all the $f = (f_0, f_+, f_-)$ for which either $f_+ \neq 0, f_- = 0$ or $f_+ = 0, f_- \neq 0$. We see that for every $f = (f_0, f_+, f_-) \in V_{gen}^*$ there exists exactly one element $f' = (f_0, f'_+, f'_-) \in V_{gen}^*$ in its $G_{n,\mu}$-orbit such that $|f'_+| = |f'_-|$. 
We denote by $V_{gen}^*$ the open subset of $V_n^*$ consisting of all the $f = (f_0, f_+, f_-) \in V_0^* \times V_+^* \times V_-^*$ for which $f_+ \neq 0$ and $f_- \neq 0$.

The subset $V_{sin}^*$ consists of all the $f = (f_0, f_+, f_-)$ for which either $f_+ \neq 0, f_- = 0$ or $f_+ = 0, f_- \neq 0$.

We see that for every $f = (f_0, f_+, f_-) \in V_{gen}^*$ there exists exactly one element $f' = (f_0, f'_+, f'_-)$ in its $G_{n,\mu}$-orbit such that $|f'_+| = |f'_-|$.

In the same way, for $f = (f_0, f_+, 0)$ (resp. $f = (f_0, 0, f_-)$) $\in V_{sin}^*$, there exists exactly one element $f' = (f_0, f'_+, 0)$ (resp. $f' = (f_0, 0, f'_-)$) in its $G_{n,\mu}$-orbit for which $|f'_+| = 1$ (resp. $|f'_-| = 1$).
Let

\[ D = \{ (f_0, f_+, f_-) : |f_+| = |f_-| \neq 0 \}, \]
\[ S_+ = \{ (f_0, f_+, 0) : |f_+| = 1 \}, \]
\[ S_- = \{ (f_0, 0, f_-) : |f_-| = 1 \}, \]
and
\[ S = S_+ \cup S_. \]
We assume for simplicity that $V_{gen}^* \neq \emptyset$, i.e., $\mu_+ \neq \emptyset$ and $\mu_- \neq \emptyset$. 
We assume for simplicity that $V^*_{gen} \neq \emptyset$, i.e., $\mu_+ \neq \emptyset$ and $\mu_- \neq \emptyset$. The orbit space $g^{*,\mu}_{n,\mu}/G_{n,\mu}$ can then be written as the disjoint union $\Gamma$ of the sets

\begin{align*}
\Gamma_0 & = \mathbb{R} \times V_0^*, \text{ corresponding to the unitary characters of } G_{n,\mu}, \\
\Gamma_1 & = \mathcal{S} \simeq V_{sin}^*/G_{n,\mu}, \\
\Gamma_2 & = \mathcal{D} \simeq V_{gen}^*/G_{n,\mu}, \\
\Gamma_3 & = \{+,-\} \simeq \{\Omega_+,\Omega_-\}/G_{n,\mu}.
\end{align*}
Theorem

1. A properly converging sequence \((\Omega f_k)_k\) with 
   \(f_k = (f_{k,0}, f_{k,+}, f_{k,-}) \in \mathcal{D}\) has either a unique limit point \(\Omega_f\) for 
   some \(f \in \mathcal{D}\) and then \(f = \lim_k f_k\), or \(\lim_k (f_{k,+}, f_{k,-}) = 0\) and 
   then the limit set \(L\) of the sequence is given by 
   \[
   L = \{ \Omega(f_{0,+},0), \Omega(f_{0},0,-), \mathbb{R} \},
   \]

   where \(f_0 = \lim_k f_{k,0}\), \(f_+ = \lim_k r(f_{k,+}) \cdot f_{k,+} \in \mathcal{S}_+\) and 
   \(f_- = \lim_k q(f_{k,-}) \cdot f_{k,-} \in \mathcal{S}_-\).

2. A properly converging sequence \((\Omega f_k)_k\) with 
   \(f_k = (f_{k,0}, f_{k,+}, f_{k,-}) \in \mathcal{S}\) has the limit set 
   \[
   L = \{ \Omega_f, \mathbb{R} \},
   \]

   where \(f = \lim_k f_k \in \mathcal{S}\).
Corollary

The orbit $\Omega_f$ for $f \in D$ is closed in $g^*_{n,\mu}$. The closure of the orbit $\Omega_f$ for $f \in S$ is the set $\{\Omega_f, \mathbb{R}\}$.
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Corollary

For $\varepsilon \in \{+, -\}$, the boundary of the open orbit $\Omega_\varepsilon$ is the subset $\mathbb{R} \times V_n^* \times \{0\} = Z^\perp \simeq g_{V_n}^*$. 
On the other hand, for every coadjoint orbit we can write down a corresponding irreducible representation as an induced representation by using Kirillov’s orbit theory.
On the other hand, for every coadjoint orbit we can write down a corresponding irreducible representation as an induced representation by using Kirillov’s orbit theory.
Let $P = \exp(\sum_{j=1}^{n} \mathbb{R} Y_j + \mathbb{R} Z)$. This is a closed connected normal abelian subgroup of $G_{n,\mu}$. Let also $\xi := \sum_{j=1}^{n} \mathbb{R} X_j$ and $\eta := \sum_{j=1}^{n} \mathbb{R} Y_j \subset V_n$ (an abelian subalgebra of $g_{n,\mu}$), then $X := \exp(\xi)$ and $Y = \exp(\eta)$ are closed connected abelian subgroups of $G_{n,\mu}$. We have

$$G_{n,\mu} = \exp(\mathbb{R} A) \cdot X \cdot P = S \cdot P,$$

where $S := \exp(\mathbb{R} A) \cdot X$ is a subgroup of $G_{n,\mu}$. The irreducible representations $\pi_{\varepsilon}, \varepsilon = \pm$, corresponding to the orbits $\Omega_{\varepsilon}$ are of the form

$$\pi_{\varepsilon} := \text{ind}_{P}^{G_{n,\mu}} \chi_{\varepsilon} Z^*.$$
The Hilbert space of $\pi_\varepsilon$ is $L^2(S_n) = L^2(\mathbb{R}^{n+1})$. 
The Hilbert space of $\pi_\varepsilon$ is $L^2(S_n) = L^2(\mathbb{R}^{n+1})$.
For $F \in L^1(G_{n,\mu})$ and $\xi \in L^2(G_{n,\mu}/P_n)$, the operator $\pi_\varepsilon(F)$ is given by the kernel function

$$F_\varepsilon((a', x'), (a, x)) = \hat{F}^p_n\left((a' - a, a \cdot (x' - x)); (-\varepsilon \sum_{j=1}^{n} e^{(\lambda_j - 2)a_j Y_j^*}, \varepsilon e^{-2a})\right) e^{|\lambda|a}.$$
For \( v^* \in V_n^* \), the irreducible representation \( \pi_{v^*} \) on \( L^2(\mathbb{R}) \) is defined by

\[
\pi_{v^*} := \text{ind}_{H_n}^{G_n, \mu} \chi_{v^*},
\]

where \( H_n := \exp(\mathfrak{h}_n) \).
For $v^* \in V_n^*$, the irreducible representation $\pi_{v^*}$ on $L^2(\mathbb{R})$ is defined by

$$\pi_{v^*} := \text{ind}_{H_n}^{G_n, \mu} \chi_{v^*},$$

where $H_n := \exp(\mathfrak{h}_n)$.

The kernel function $F_{v^*}$ of the operator $\pi_{v^*}(F), F \in L^1(G_n, \mu)$, is given by

$$F_{v^*}(a, b) = \widehat{F}^{\mathfrak{h}_n}(a - b, b \cdot v^*, 0) \quad \text{for} \quad a, b \in \mathbb{R}. \quad (9)$$
For $v^* \in V_n^*$, the irreducible representation $\pi_{v^*}$ on $L^2(\mathbb{R})$ is defined by

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$$F_{v^*}(a, b) = \hat{F}_{\mathfrak{h}_n}(a - b, b \cdot v^*, 0) \quad \text{for} \quad a, b \in \mathbb{R}. \quad (9)$$

Finally, for $(a^*, v_0^*) \in \mathbb{R} \times V_0^*$ we have the unitary characters

$$\chi(a^*, v_0^*)(a, v_0, v, c) := e^{-2\pi i(a^*a + v_0^*(v_0))} \quad \text{for} \quad a, c \in \mathbb{R}, \, v_0 \in V_0, \, v \in V_1.$$
Almost $C_0(\mathcal{K})$-C*-algebras

Definition

Let $A$ be a C*-algebra and $\hat{A}$ be the spectrum of $A$.

1. Suppose that the spectrum $\hat{A} = \bigcup_{i=0}^{d} \Gamma_i$ is Hausdorff of step $d$. Furthermore we assume that for every $i \in \{0, \cdots, d\}$ there exists a Hilbert space $\mathcal{H}_i$ and that a concrete realization $(\pi_\gamma, \mathcal{H}_i)$ of $\gamma$ on the Hilbert space $\mathcal{H}_i$ for every $\gamma \in \Gamma_i$ is chosen.
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The set $S_0$ is the collection $\mathcal{X}$ of all characters of $A$. 
Almost $C_0(\mathcal{K})$-$C^*$-algebras

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The set $S_0$ is the collection $\mathcal{X}$ of all characters of $A$.

2. For a subset $S \subset \hat{A}$, denote by $CB(S)$ the *-algebra of all uniformly bounded operator fields 

\[(\psi(\gamma) \in \mathcal{B}(\mathcal{H}_i))_{\gamma \in S \cap \Gamma_i, i = 1, \cdots, d},\]

which are operator norm continuous on the subsets $\Gamma_i \cap S$ for every $i \in \{1, \cdots, d\}$ for which $\Gamma_i \cap S \neq \emptyset$. We provide the *-algebra $CB(S)$ with the infinity-norm:

\[\|\psi\|_S := \sup_{\gamma \in S} \|\psi(\gamma)\|_{op}.\]
Definition
A C*-algebra $A$ is said to be almost $C_0(K)$ if for every $a \in A$:

1. The mappings $\gamma \mapsto \mathcal{F}(a)(\gamma)$ are norm continuous on the different sets $\Gamma_i$, where $\mathcal{F} : A \to l^\infty(\hat{A})$ is the Fourier transform given by

$$\mathcal{F}(a)(\gamma) = \hat{a}(\gamma) := \pi_\gamma(a) \quad \text{for} \quad \gamma \in \hat{A} \quad \text{and} \quad a \in A.$$
Definition
A C*-algebra $A$ is said to be *almost* $C_0(K)$ if for every $a \in A$:

1. The mappings $\gamma \mapsto F(a)(\gamma)$ are norm continuous on the different sets $\Gamma_i$, where $F : A \to l^\infty(\hat{A})$ is the Fourier transform given by

   $$F(a)(\gamma) = \hat{a}(\gamma) := \pi_{\gamma}(a) \quad \text{for} \quad \gamma \in \hat{A} \text{ and } a \in A.$$  

2. For each $i = 1, \cdots, d$, we have a sequence $(\sigma_{i,k} : CB(S_{i-1}) \to CB(S_i))_k$ of linear mappings which are uniformly bounded in $k$ (and independent of $a$) such that

   $$\lim_{k \to \infty} \text{dis}\left((\sigma_{i,k}(F(a)|_{S_{i-1}}) - F(a)|_{\Gamma_i}), C_0(\Gamma_i, K(H_i))\right) = 0,$$

   and

   $$\lim_{k \to \infty} \text{dis}\left((\sigma_{i,k}(F(a^*)|_{S_{i-1}}) - F(a^*)|_{\Gamma_i}), C_0(\Gamma_i, K(H_i))\right) = 0.$$
Definition
Let $D^*(A)$ be the set of all operator fields $\varphi$ defined over $\hat{A}$ such that

1. The field $\varphi$ is uniformly bounded, i.e., we have that $\|\varphi\| := \sup_{\gamma \in \hat{A}} \|\varphi(\gamma)\|_{op} < \infty$.
2. $\varphi|_{\Gamma_i} \in CB(\Gamma_i)$ for every $i = 0, 1, \ldots, d$. 
Definition
Let \( D^*(A) \) be the set of all operator fields \( \varphi \) defined over \( \hat{A} \) such that

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   \[ \|\varphi\| := \sup_{\gamma \in \hat{A}} \|\varphi(\gamma)\|_{op} < \infty. \]

2. \( \varphi|_{\Gamma_i} \in CB(\Gamma_i) \) for every \( i = 0, 1, \ldots, d \).

3. For every sequence \( (\gamma_k)_{k \in \mathbb{N}} \) going to infinity in \( \hat{A} \), we have that
   \( \lim_{k \to \infty} \|\varphi(\gamma_k)\|_{op} = 0. \)
Definition
Let $D^*(A)$ be the set of all operator fields $\varphi$ defined over $\hat{A}$ such that

1. The field $\varphi$ is uniformly bounded, i.e., we have that
$$
\|\varphi\| \defeq \sup_{\gamma \in \hat{A}} \|\varphi(\gamma)\|_{op} < \infty.
$$

2. $\varphi|_{\Gamma_i} \in CB(\Gamma_i)$ for every $i = 0, 1, \ldots, d$.

3. For every sequence $(\gamma_k)_{k \in \mathbb{N}}$ going to infinity in $\hat{A}$, we have that
$$
\lim_{k \to \infty} \|\varphi(\gamma_k)\|_{op} = 0.
$$

4. For each $i = 1, 2, \ldots, d$,

$$
\lim_{k \to \infty} \text{dis}
\left(
\left(\sigma_{i,k}(\varphi|_{S_{i-1}}) - \varphi|_{\Gamma_i}\right),
C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i))
\right) = 0
$$

and

$$
\lim_{k \to \infty} \text{dis}
\left(
\left(\sigma_{i,k}(\varphi^*|_{S_{i-1}}) - (\varphi|_{\Gamma_i})^*\right),
C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i))
\right) = 0.
$$
We see immediately that if \( A \) is almost \( C_0(\mathcal{K}) \), then for every \( a \in A \), the operator field \( \mathcal{F}(a) \) is contained in the set \( D^*(A) \). In fact it turns out that \( D^*(A) \) is a C*-subalgebra of \( l_\infty(\hat{A}) \) and that \( A \) is isomorphic to \( D^*(A) \).
We see immediately that if $A$ is almost $C_0(\mathcal{K})$, then for every $a \in A$, the operator field $\mathcal{F}(a)$ is contained in the set $D^*(A)$. In fact it turns out that $D^*(A)$ is a $C^*$-subalgebra of $l^\infty(\hat{A})$ and that $A$ is isomorphic to $D^*(A)$.

**Theorem**

(Inoue-Lin-L.) Let $A$ be a separable $C^*$-algebra which is almost $C_0(\mathcal{K})$. Then the subset $D^*(A)$ of the $C^*$-algebra $l^\infty(\hat{A})$ is a $C^*$-subalgebra which is isomorphic to $A$ under the Fourier transform.
Theorem

(Inoue-Lin-L.) The C*-algebra of $G_{n,\mu}$ is an almost $C_0(\mathcal{K})$-C*-algebra. In particular, the Fourier transform maps $C^*(G_{n,\mu})$ onto the subalgebra $D^*(G_{n,\mu})$ of $l^\infty(\hat{G}_{n,\mu})$. 
Thank you