The C^*-algebras of some exponential Lie groups

Göteborg, August 15 2016

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Let A be an involutive Banach algebra.
The C^*-algebra of A is the completion $C^*(A)$ of A with respect to the C^*-norm

$$\|a\|_{C^*} := \sup_{\pi \in \mathrm{Rep}} \|\pi(a)\|_{\mathrm{op}}$$

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where

$$\int_G \Delta_G(t^{-1}) f(t^{-1}) dt = \int_G f(t) dt, f \in L^1(G).$$

How to describe

$$C^*(G) := C^*(L^1(G))?$$

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Then $\mathcal{F}(a)$ is contained in the algebra of all bounded operator fields over $\widehat{A}$

$$l^\infty(\widehat{A}) = \left\{ \phi = \left(\phi(\pi_\gamma) \in \mathcal{B}(\mathcal{H}_\gamma)\right)_{\gamma \in \widehat{A}} \mid \|\phi\|_\infty := \sup_{\gamma \in \widehat{A}} \|\phi(\pi_\gamma)\|_{op} < \infty \right\}$$

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and the mapping

$$\mathcal{F} : A \to l^\infty(\widehat{A}),\ a \mapsto \hat{a}$$

is an isometric $*$-homomorphism.

Examples:

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  Here

$$C_0(\widehat{G}) = \{\varphi = (\varphi(\pi) \in \mathcal{B}(\mathcal{H}_\pi))_{\pi \in \widehat{G}} | \lim_{\pi \to \infty} \|\varphi(\pi)\|_{\mathrm{op}} = 0\}.$$

Nilpotent and exponential Lie groups

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Definition

A polynomial (resp. analytic) vector group is a real finite dimensional vector space V equipped with a polynomial (resp. analytic) group multiplication $\cdot$ such that

$$sX \cdot tX = sX + tX = (s+t)X, X \in V, s, t \in \mathbb{R}.$$

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Polynomial (resp. analytic) means for any basis $\mathcal{B} = \{Z_1, \cdots, Z_n\}$ of V there exists polynomial (resp. analytic) functions $m_j : V \times V \to V$ such that

$$X \cdot Y = \sum_{j=1}^{n} m_j(X, Y) Z_j, \; X, Y \in V.$$

Proposition

Every polynomial resp. analytic vector group is nilpotent (resp. analytic) and every connected simply connected nilpotent (resp. analytic) Lie group is isomorphic to some polynomial (resp. analytic) vector group.

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The Lie algebra of the polynomial vector group $(V, \cdot)$ *can be identified with* V *and then*

$$\begin{aligned} & X \cdot Y \\ = \; & X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] + \dots \end{aligned}$$

Let $\lambda_i \in \mathbb{R}^* + i\mathbb{R}, i = 1, \cdots, n$ and let

$$T_n(\lambda_1, \cdots, \lambda_n) :=$$
$$\left\{ \begin{pmatrix} e^{t_1\lambda_1} & \alpha_{1,2} & \cdots & \cdots & \alpha_{1,n} \\ 0 & e^{t_2\lambda_2} & \alpha_{2,3} & \ddots\ddots & \alpha_{2,n} \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \alpha_{n-1,n} \\ 0 & 0 & \cdots & 0 & e^{t_n\lambda_n} \end{pmatrix}, t_i \in \mathbb{R},\ \alpha_{i,j} \in \mathbb{C} \right\}.$$

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Theorem

Every exponential Lie group is isomorphic to a closed connected subgroup of some $T_n(\lambda_1, \cdots, \lambda_n)$.

Kirillov theory

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Let $G = \exp(\mathfrak{g})$ be an exponential Lie group.
Let $\mathfrak{g}^*$ be the algebraic dual of $\mathfrak{g}$. The group G acts by conjugation on G

$$
\begin{aligned}
& t \mapsto sts^{-1}, s, t \in G, \\
& \Rightarrow \mathrm{Ad}\,(\mathrm{g})\mathrm{X} = \frac{\mathrm{d}}{\mathrm{dt}}\mathrm{g}\exp(\mathrm{tX})\mathrm{g}^{-1}, \mathrm{X} \in \mathfrak{g}, \mathrm{g} \in \mathrm{G} \\
& \Rightarrow \mathrm{Ad}^*(\mathrm{g})\ell = \mathrm{Ad}\,(\mathrm{g}^{-1})^{\mathrm{tr}}\ell, \mathrm{i.e.} \\
& \langle \mathrm{Ad}^*(\mathrm{g})\ell, \mathrm{X}\rangle = \langle \ell, \mathrm{Ad}\,(\mathrm{g}^{-1})\mathrm{X}\rangle, \ell \in \mathfrak{g}^*, \mathrm{X} \in \mathfrak{g}.
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For any $\ell \in \mathfrak{g}^*$ let

$$
\begin{aligned}
G(\ell) &= \{g \in G; \mathrm{Ad}^*(\mathrm{g})\ell = \ell\} \\
&= \exp(\{S \in \mathfrak{g}; \langle \ell, [S, \mathfrak{g}]\rangle = \{0\}) \\
&= \exp(\mathfrak{g}(\ell)).
\end{aligned}
$$

Let

$$\Omega_\ell := \{\, \mathrm{Ad}^*(\mathrm{g})\ell; \mathrm{g} \in \mathrm{G}\} \simeq \mathrm{G}/\mathrm{G}(\ell)$$

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Then Ω_ℓ is a locally closed subset and a smooth submanifold of $\mathfrak{g}^*$ of dimension $\dim(\mathfrak{g}/\mathfrak{g}(\ell))$.
If G is nilpotent then Ω_ℓ is a Zarisky closed subset of g^*.

Kirillov theory

A polarization at ℓ is a subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ such that

$$\langle \ell, [\mathfrak{p}, \mathfrak{p}] \rangle = \{0\},$$
$$\dim(\mathfrak{p}) = \frac{\dim(\mathfrak{g}/\mathfrak{g}(\ell))}{2} + \dim(\mathfrak{g}(\ell)).$$

Let $P = \exp(\mathfrak{p})$ and

$$\chi_\ell(\exp(X)) = e^{-i\langle \ell, X \rangle}, X \in \mathfrak{p}.$$

Then χ_ℓ is a unitary character of the group P.

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We say that $\mathfrak{p}$ is a Pukanszky (Puk) representation at ℓ if $\mathrm{Ad}^*(\mathrm{P})\ell = \ell + \mathfrak{p}^\perp$.

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$$\pi_{\ell,\mathfrak{p}} = \mathrm{ind}_P^G \chi_\ell$$

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$$\begin{aligned} & L^2(G/P, \chi_\ell) \\ := \; & \{\xi : G \to \mathbb{C}; \xi(gp) = \chi_\ell(p^{-1})\left(\frac{\Delta_P(p)}{\Delta_G(p)}\right)^{1/2}\xi(g), g \in G, p \in P; \\ & \xi \text{ measurable }, \|\xi\|_2^2 := \oint_{G/P} |\xi(g)|^2 d\dot{g} < \infty\} \\ \simeq \; & L^2(\mathbb{R}^m), (m = \dim(\mathfrak{g}/\mathfrak{p}). \end{aligned}$$

Kirillov theory

Theorem

(Kirillov) (G nilpotent) For every $\pi \in \widehat{G}$ and every $c \in C^*(G)$ the operator $\pi(c)$ is compact and for every $f \in \mathcal{S}(G)$ the operator $\pi(f)$ is trace class.

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Theorem

(Kirillov , Bernat, Pukanszky, Vergne) Let $G = exp(\mathfrak{g})$ be an exponential Lie group.

- *For every $\ell \in \mathfrak{g}^*$ there exists a Puk-polarization at ℓ.*
- *For every $\ell \in \mathfrak{g}^*$ and for every Puk-polarization $\mathfrak{p}$ at ℓ the representation $\pi_{\ell,\mathfrak{p}}$ is irreducible.*
- *For $\ell, \ell' \in \mathfrak{g}^*$ for any Puk-polarization $\mathfrak{p}$ at ℓ and $\mathfrak{p}'$ at ℓ' we have that*

$$\pi_{\ell,\mathfrak{p}} \simeq \pi_{\ell',\mathfrak{p}'} \Leftrightarrow \Omega_\ell = \Omega_{\ell'}.$$

- *every $\pi \in \widehat{G}$ is equivalent to some $\pi_{\ell,\mathfrak{p}}$.*

Kirillov theory

Hence the mapping

$$\mathcal{K} : \mathfrak{g}^*/G \to \widehat{G}; \Omega_\ell \mapsto [\pi_{\ell,\mathfrak{p}}] = [\pi_\ell]$$

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Theorem

(Kirillov-Brown,Fujiwara, Leptin-L.) The mapping $\mathcal{K}$ is a homeomorphism.

An example: The group $G_{5,6}$

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The Lie algebra $\mathfrak{g}_{5,6}$ is spanned by the basis $\mathcal{B} = \{A, B, C, U, V\}$ equipped with the Lie brackets

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The group $G_{5,6} = \exp(\mathfrak{g}_{5,6})$ can be realized as $\mathbb{R}^5$ with the multiplication

$$\begin{aligned} & (a, b, c, u, v) \cdot (a', b', c', u', v') \\ = \; & (a + a', b + b', c + c' - a'b, u + u' - a'c + \frac{a'^2 b}{2}, \\ & v + v' - a'u + \frac{bc'}{2} - \frac{b'c}{2} + \frac{a'bb'}{2} \\ & + \frac{a'^2 c}{2} - \frac{a'^3 b}{6}). \end{aligned} \tag{1}$$

We use the Euclidean scalar product on $\mathfrak{g}_{5,6}$ to identify $\mathfrak{g}_{5,6}^*$ with $\mathfrak{g}_{5,6} = \mathbb{R}^5$ and we we obtain the following expression for $\mathrm{Ad}^*(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{u}, \mathrm{v})$:

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$$
\begin{aligned}
& \mathrm{Ad}^*((\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{u}, \mathrm{v}))(\alpha, \beta, \rho, \mu, \nu) \\
= \quad & (\alpha + \rho b + \mu c - \mu \frac{ab}{2} + \nu u - \nu \frac{b^2}{2} - \nu \frac{ac}{2} + \nu \frac{a^2 b}{6}, \\
& \beta - \rho a + \mu \frac{a^2}{2} + \nu c + \nu \frac{ab}{2} - \nu \frac{a^3}{6}, \\
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3) The generic orbits: Let $\nu \neq 0$. The orbit $\mathcal{O}_\nu$ of the element $\ell_\nu = (0, 0, 0, 0, \nu)$ is given by:

$$\mathcal{O}_\nu \quad = \quad \{(a, b, c, u, \nu),\ a, b, c, u \in \mathbb{R}\}.$$

We denote by $\Gamma_3^{5,6}$ the orbit space of this layer and we parametrize it by

$$\Gamma_3^{5,6} := \{\mathcal{O}_\nu \equiv \nu, \nu \in \mathbb{R}^*\}. \tag{2}$$

2) Let for $(\beta, \mu) \in \mathbb{R} \times \mathbb{R}^*$

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$$\mathcal{O}_\sigma = \mathcal{O}_{\ell_\sigma} = \{(a, b, \sigma, 0, 0) | a, b \in \mathbb{R}\}.$$

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$$\Gamma_0^{5,6} := \{(\alpha, \beta, 0, 0, 0) | \alpha, \beta \in \mathbb{R}\}.$$

Theorem

Let $\overline{\mathcal{O}} = (\mathcal{O}_{\nu_k})_k \subset \Gamma_3^{5,6}$ be a sequence, such that $\lim_{k\to\infty} \nu_k = 0$. If the sequence $\overline{\mathcal{O}}$ is properly converging then

$$\begin{aligned} L(\overline{\mathcal{O}}) &= \Gamma_2^{5,6} \cup \Gamma_1^{5,6} \cup \Gamma_0^{5,6} \\ &= V^{\perp}. \end{aligned}$$

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Then

$$\text{sign}(\mu_k) \text{ is constant and } \lim_{k\to\infty} \beta_k \mu_k = \kappa \text{ exists.}$$

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If $\kappa = 0$ and $\nu_k > 0, k \in \mathbb{N}$, then $\beta_\infty = \lim_{k\to\infty} \beta_k$ exists in $[-\infty, \infty[$ and $L(\overline{\mathcal{O}}) = \mathbb{R}A^* + [\beta_\infty, \infty[B^*$.

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Let $\overline{\mathcal{O}} = (\overline{\mathcal{O}}_{\sigma_k})_k \subset \Gamma_1^{5,6}$ be a properly converging sequence with $\lim\limits_{k\to\infty} \sigma_k = 0$. Then $L(\overline{\mathcal{O}}) = \mathbb{R}A^* + \mathbb{R}B^*$.

An example: The C^*-algebra of the Heisenberg group

Let $H_n = \mathbb{R}^{2n+1}$ on which the multiplication is given by

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + \frac{1}{2}(x \cdot y' - x' \cdot y)),$$

where $x \cdot y = x_1 y_1 + \cdots + x_n y_n$ denotes the Euclidean scalar product on $\mathbb{R}^n$.

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where $x \cdot y = x_1 y_1 + \cdots + x_n y_n$ denotes the Euclidean scalar product on $\mathbb{R}^n$.
The centre of H_n is the subgroup $\mathcal{Z} := \{0_n\} \times \{0_n\} \times \mathbb{R}$ and the commutator subgroup $[H_n, H_n]$ of H_n is given by $[H_n, H_n] = \mathcal{Z}$.

The Lie algebra $\mathfrak{g}$ of H_n has the basis

$$\mathcal{B} := \{X_j, Y_j, j = 1 \cdots, n, Z = (0_n, 0_n, 1)\},$$

where $X_j = (e_j, 0_n, 0)$, $Y_j = (0_n, e_j, 0), j = 1, \cdots, n$ and e_j is the j'th canonical basis vector of $\mathbb{R}^n$, with the non trivial brackets

$$[X_i, Y_j] = \delta_{i,j} Z.$$

The unitary dual of H_n.

The infinite dimensional irreducible representations:
For every $\lambda \in \mathbb{R}^*$, there exists a unitary representation π_λ of H_n on the Hilbert space $L^2(\mathbb{R}^n)$, which is given by the formula

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$$\pi_\lambda(x, y, t)\xi(s) := e^{-2\pi i\lambda t - 2\pi i \frac{\lambda}{2} x\cdot y + 2\pi i \lambda s\cdot y}\xi(s - x),$$
$$s \in \mathbb{R}^n, \xi \in L^2(\mathbb{R}^n), (x, y, t) \in H_n.$$

The unitary dual of H_n.

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The representation π_λ is equivalent to the induced representation $\tau_\lambda := \mathrm{ind}_P^{H_n}\chi_\lambda$, where $P = \{0_n\} \times \mathbb{R}^n \times \mathbb{R}$ is a polarization at the linear functional $\ell_\lambda((x,y,t)) := \lambda t, (x,y,t) \in \mathfrak{g}$ and where χ_λ is the character of P defined by $\chi_\lambda(0_n,y,t) = e^{-2\pi i\lambda t}$.

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The theorem of Stone-Von Neumann (1929) tells us that every infinite dimensional unitary representation of H_n is equivalent to one of the π_λ's.

The finite dimensional irreducible representations

Since H_n is nilpotent, every irreducible finite dimensional representation of H_n is one-dimensional, by Lie's theorem. Any one-dimensional representation is a unitary character $\chi_{a,b}$, $(a, b) \in \mathbb{R}^n \times \mathbb{R}^n$, of H_n, which is given by

$$\chi_{a,b}(x, y, t) = e^{-2\pi i(a \cdot x + b \cdot y)}, (x, y, t) \in H_n.$$

The Fourier transform

Definition

Define for $c \in C^*(H_n)$ the *Fourier transform* $F(c)$ of c by

$$\hat{c}(\lambda) = \hat{c}(\pi_\lambda) = F(c)(\lambda) := \pi_\lambda(c) \in B(L^2(\mathbb{R}^n)), \lambda \in \mathbb{R}^*$$

and

$$\hat{c}(u) = F(c)(u) := \chi_u(c),\ \ u \in \mathbb{R}^{2n}.$$

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Define for $c \in C^*(H_n)$ the function

$$\hat{c}(0)(u) := \hat{c}(u), u \in \mathbb{R}^{2n}.$$

Then the mapping

$$C^*(H_n) \to C_0(\mathbb{R}^{2n}); c \mapsto \hat{c}(0),$$

is a surjective homomorphism.

The topology of $\widehat{C^*(H_n)}$

As for the topology of the dual space, it is well known that $[\pi_\lambda]$ tends to $[\pi_\nu]$ in $\widehat{H_n}$ if and only if λ tends to ν in $\mathbb{R}^*$, where $[\pi]$ denotes the unitary equivalence class of the unitary representation π. Furthermore, if λ tends to 0, then the representations π_λ converge in the dual space topology to all the characters $\chi_{a,b}, a, b \in \mathbb{R}^n$.

$\lim_{\lambda \to 0} \pi_\lambda$

Choose a Schwartz-function η in $\mathcal{S}(\mathbb{R}^n)$ with L^2-norm equal to 1. For $u = (a, b)$ in $\mathbb{R}^n \times \mathbb{R}^n$, $\lambda \in \mathbb{R}^*$, we define the function $\eta(\lambda, a, b)$ by

$$\eta(\lambda, a, b)(s) := |\lambda|^{n/4} e^{2\pi i a \cdot s} \eta(|\lambda|^{1/2}(s + \frac{b}{\lambda})), \ s \in \mathbb{R}^n, \tag{3}$$

and let $\eta_\lambda(s) = |\lambda|^{n/4}\eta(|\lambda|^{1/2} s), \ s \in \mathbb{R}^n$.

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Let us compute

$$
\begin{aligned}
& \langle \pi_\lambda(x, y, t)\eta(\lambda, u), \eta(\lambda, u)\rangle \\
= \quad & \int_{\mathbb{R}^n} e^{-2\pi i\lambda t - 2\pi i(\lambda/2)x\cdot y} e^{2\pi i\lambda s\cdot y} \eta(\lambda, u)(s - x) \\
& \overline{\eta(\lambda, u)(s)} ds \\
= \quad & e^{-2\pi i\lambda t - 2\pi i\frac{\lambda}{2}x\cdot y} e^{-2\pi i a\cdot x - 2\pi i b\cdot y} \\
& \int_{\mathbb{R}^n} e^{2\pi i(\mathrm{sign}\lambda)|\lambda|^{1/2} s\cdot y} \eta(s - |\lambda|^{1/2}x)\overline{\eta(s)} ds \\
\to \quad & e^{-2\pi i a\cdot x - 2\pi i b\cdot y} \int_{\mathbb{R}^n} \eta(s)\overline{\eta(s)} ds = e^{-2\pi i a\cdot x - 2\pi i b\cdot y}.
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\to \quad & e^{-2\pi i a\cdot x - 2\pi i b\cdot y}\int_{\mathbb{R}^n}\eta(s)\overline{\eta(s)}ds = e^{-2\pi i a\cdot x - 2\pi i b\cdot y}.
\end{aligned}$$

It follows also that the convergence of the coefficients to the characters $\chi_{a,b}$ is uniform in u and uniform on compacta.

Properties of the Fourier transform

Let us compute for $f \in L^1(H_n)$ the operator $\pi_\lambda(f)$. We have for $\xi \in L^2(\mathbb{R}^n)$ and $s \in \mathbb{R}^n$ that

$$\pi_\lambda(f)\xi(s) \;=\; \int_{\mathbb{R}^n} \hat{f}^{2,3}(s-x, -\frac{\lambda}{2}(s+x), \lambda)\xi(x)dx.$$

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Here

$$\hat{f}^{2,3}(s,u,\lambda) = \int_{\mathbb{R}^n \times \mathbb{R}} f(s,y,t)e^{-2\pi i(y\cdot u+\lambda t)}dydt, \ (s,u,\lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$$

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If we take now a Schwartz-functions $f \in \mathcal{S}(H_n)$, then the operator $\pi_\lambda(f)$ is Hilbert-Schmidt and its Hilbert-Schmidt norm $\|\pi_\lambda(f)\|_{\text{H.S.}}$ is given by

$$\begin{aligned} \|\pi_\lambda(f)\|^2_{\text{H.S.}} &= \int_{\mathbb{R}^2} |f_\lambda(s,x)|^2 dxds = \int_{\mathbb{R}^2} |\hat{f}^{2,3}(s,\lambda x,\lambda)|^2 dsdx \\ &< \infty. \end{aligned}$$

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Proposition

For any $c \in C^*(H_n)$ and $\lambda \in \mathbb{R}^*$, the operator $\pi_\lambda(c)$ is compact , the mapping $\mathbb{R}^* \to B(L^2(\mathbb{R}^n)) : \lambda \mapsto \pi_\lambda(c)$ is norm continuous and tending to 0 for λ going to infinity.

The condition in 0 and the mappings σ_λ

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Recall that for $\eta \in \mathcal{S}(\mathbb{R}^n)$, $\|\eta\|_2 = 1$, $u = (a, b) \in \mathbb{R}^n \times \mathbb{R}^n$, $\lambda \in \mathbb{R}^*$, we have defined

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Simple calculation shows that

$$\pi_\lambda(f) = \frac{1}{|\lambda|^n} \int_{\mathbb{R}^{2n}} \pi_\lambda(f) \circ P_{\eta(\lambda,u)} du$$

for any $f \in S(H_n)$, where $P_{\eta(\lambda,u)}$ is the orthogonal projection onto the one dimensional subspace $\mathbb{C}\eta(\lambda, u)$.

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Definition

Define for $\lambda \in \mathbb{R}^*$ and $h \in C_0(\mathbb{R}^{2n})$ the linear operator

$$\sigma_\lambda(h) \; := \; \frac{1}{|\lambda|^n} \int_{\mathbb{R}^{2n}} h(u) P_{\eta(\lambda,u)} du.$$

Theorem

(L.,Turowska 2008)

1. $\sigma_\lambda(h)$ *is compact and* $\|\sigma_\lambda(h)\|_{op} \leq \|h\|_\infty$.
2. $\sigma_\lambda(h^*)(\lambda) = \sigma_\lambda(h)^*, h \in C_0(\mathbb{R}^{2n})$.
3. $\|\pi_\lambda(a) - \sigma_\lambda(F(a)(0))\|_{op} \to 0$ *as* $\lambda \to 0$ *for each* $a \in C^*(H_n)$.

Definition

Let $D_\nu^*(H_n)$ be the space consisting of all the fields $(F(\lambda))_{\lambda\in\mathbb{R}}$, such that

1. $F(\lambda)$ is a compact operator on $L^2(\mathbb{R}^n)$ for every $\lambda \in \mathbb{R}^*$,
2. $F(0) \in C^*(\mathbb{R}^2)$,
3. the mapping $\mathbb{R}^* \to B(L^2(\mathbb{R}^n)) : \lambda \mapsto F(\lambda)$ is norm continuous,
4. $\lim_{\lambda\to\infty} \|F(\lambda)\|_{\mathrm{op}} = 0$,
5. $\lim_{\lambda\to 0} \|F(\lambda) - \sigma_\lambda(F(0)\|_{\mathrm{op}} = 0$.

We have the following characterisation of $C^*(H_n)$.

Theorem

(L., Turowska) The space $D^*_\nu(H_n)$ is a C^*-algebra which is isomorphic to $C^*(H_n)$ under the Fourier transform.

Norm controlled dual limits

Theorem

(I. Beltita, B. Beltita, J. L.) Let $\mathcal{A}$ *be a separable liminary* C^**-algebra, let* $\overline{\pi} = \{[(\pi_k, \mathcal{H}_k)]\}_{k\in\mathbb{N}}$ *be a properly converging sequence in* $\widehat{\mathcal{A}}$ *with limit set* L*.*

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There exists a uniformly bounded sequence of bounded linear mappings

$$\tilde{\sigma}_k : \mathcal{F}(\mathcal{A})_{|L} \to \mathcal{B}(\mathcal{H}_k)$$

such that

$$\lim_{k\to\infty} \|\tilde{\sigma}_k(\hat{a}_{|L}) - \pi_k(a)\|_{\mathrm{op}} = 0, a \in \mathcal{A}.$$

Conclusion

The problem is how to find a precise expression for the mappings $\tilde{\sigma}_k$, whenever the algebra $\mathcal{A}$ and the representations π_k are concretely given.

C^*-algebras with norm controlled dual limits

Definition

i) Let S be a topological space. We say that S is *locally compact of step* $\leq N$ if there exists a finite increasing family $\emptyset \neq S_0 \subset \cdots \subset S_N = S$ of closed subsets of S, such that the subsets $\Gamma_0 = S_0$ and $\Gamma_i := S_i \setminus S_{i-1}$, $i = N, \ldots, 1$, are locally compact and Hausdorff in their relative topologies.

C^*-algebras with norm controlled dual limits

ii) Let S be locally compact of step $\leq d$, and let $\{\mathcal{H}_i\}_{i=1,\dots,d}$ be Hilbert spaces. For a closed subset $M \subset S$, denote by $CB(M)$ the unital C^*-algebra of all uniformly bounded operator fields $(\psi(\gamma) \in \mathcal{B}(\mathcal{H}_i))_{\gamma \in S \cap \Gamma_i, i=1,\dots,d}$, which are operator norm continuous on the subsets $\Gamma_i \cap M$ for every $i \in \{1, \dots, d\}$ with $\Gamma_i \cap M \neq \emptyset$

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$$\|\varphi\|_M = \sup \left\{ \|\varphi(\gamma)\|_{\mathcal{B}(\mathcal{H}_i)} \mid M \cap \Gamma_i \neq \emptyset, \gamma \in M \cap \Gamma_i \right\}.$$

Spaces of operator fields with norm controlled dual limits

Definition

Let $\emptyset = S_{-1} \subset S_0 \subset \cdots \subset S_d = S$ be a locally compact topological space of step $\leq d$. Choose for every $i = 1, \ldots, d$ a Hilbert space $\mathcal{H}_i$ and assume that $\mathcal{H}_0 = \mathbb{C}$.
Let $B^*(S)$ be the set of all operator fields φ defined over S such that

- $\varphi(\gamma) \in \mathcal{K}(\mathcal{H}_i)$ for every $\gamma \in \Gamma_i = S_{i-1} \setminus S_i$, $i = 1, \ldots, d$.
- The field φ is uniformly bounded, that is, we have that

$$\|\varphi\| = \sup\left\{\|\varphi(\gamma)\|_{\mathcal{B}(\mathcal{H}_i)} \mid \gamma \in \Gamma_i,\ i = 1, \ldots, d\right\} < \infty.$$

- The mappings $\gamma \to \varphi(\gamma)$ are norm continuous on the difference sets Γ_i.

- We have for any sequence $(\gamma_k)_{k\in\mathbb{N}} \subset S$ going to infinity, that

$$\lim_{k\to\infty} \|\varphi(\gamma_k)\|_{\mathrm{op}} = 0.$$

- For any $i = 1, \dots, d+1$, and for any converging sequence contained in $\Gamma_i = S_i \setminus S_{i-1}$ with limit set outside Γ_i, there exists a properly converging sub-sequence $\overline{\gamma} = (\gamma_k)_{k\in\mathbb{N}}$, a constant $C > 0$ and for every $k \in \mathbb{N}$ an involutive linear mapping $\tilde{\sigma}_{\overline{\gamma},k} \colon CB(S_i) \to \mathcal{B}(\mathcal{H}_i)$, which is bounded by $C\|.\|_{S_i}$, such that

$$\lim_{k\to\infty} \|\varphi(\gamma_k) - \tilde{\sigma}_{\overline{\gamma},k}(\varphi|_{S_i})\|_{\mathcal{B}(\mathcal{H}_i)} = 0.$$

Theorem

Let S be a locally compact topological space of step $\leq d$. Then the set $B^*(S)$ of the preceding frame is a closed involutive subspace of $\ell^\infty(S)$. Furthermore $B^*(S)$ is a C^*-subalgebra of $\ell^\infty(S)$ with spectrum S if and only if all the mappings $\tilde{\sigma}_{\overline{\gamma},k}$ are almost homomorphisms, i.e.,

$$\lim_{k\to\infty} \|\tilde{\sigma}_{\overline{\gamma},k}(\varphi\cdot\psi) - \tilde{\sigma}_{\overline{\gamma},k}(\varphi)\cdot\tilde{\sigma}_{\overline{\gamma},k}(\psi)\|_{\mathcal{B}(\mathcal{H}_i)} = 0, \quad \varphi, \psi \in B^*(S),$$

and the restrictions $B^*(S)|_{S_{i-1}}$ contain the spaces $C_0(\Gamma_i, \mathcal{H}_i)$, $i = 1, \ldots, d+1$.

Definition

Let us denote by $\tilde{\sigma}$ the collection of all these mappings $\tilde{\sigma_{\overline{\gamma}}}$ and by

$$C_0(S, \tilde{\sigma})$$

the C^*-algebra of operator fields defined over S satisfying the conditions of the preceding theorems.

C^*-algebras with norm controlled dual limits

Definition

Let $\mathcal{A}$ be a separable liminary C^*-algebra. We say that the C^*-algebra $\mathcal{A}$ has norm controlled dual limits if A is isomorphic to some $C_0(S, \tilde{\sigma})$-C^*-algebra.

Theorem

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The C^*-algebra $C^*(G)$ has norm controlled dual limits, i.e. is isomorphic to $C^*(\widehat{G}, \tilde{\sigma})$ for some family $\tilde{\sigma}$.

An exponential example: The groups $G_{n,\mu}$.

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For

$$\mu := \{\lambda_1, \lambda_1', \cdots, \lambda_n, \lambda_n'\} \subset \mathbb{R}$$

with $\lambda_i + \lambda_i' = 2$ for all $i = 1, \cdots, n$, we define the brackets

$$[A, X_i] = \lambda_i X_i,\ [A, Y_i] = \lambda_i' Y_i,\ [A, Z] = 2Z \text{ for all } i = 1, \cdots, n,$$

and

$$[X_i, Y_j] = \delta_{i,j} Z \quad \text{for} \quad i, j = 1, \cdots, n.$$

We then obtain a structure of an exponential solvable Lie algebra on $\mathfrak{g}_{n,\mu}$, and its subalgebra $\mathfrak{h}_n$ is the Heisenberg Lie algebra. Define the diagonal operator $I_\mu : V_n \to V_n$ by

$$I_\mu(v) := \sum_i \lambda_i v_i X_i + \lambda'_i v'_i Y_i \quad \text{for} \quad v = \sum_{i=1}^n v_i X_i + \sum_{i=1}^n v'_i Y_i \in V_n.$$

We then obtain a structure of an exponential solvable Lie algebra on $\mathfrak{g}_{n,\mu}$, and its subalgebra $\mathfrak{h}_n$ is the Heisenberg Lie algebra.
Define the diagonal operator $l_\mu : V_n \to V_n$ by

$$l_\mu(v) := \sum_i \lambda_i v_i X_i + \lambda_i' v_i' Y_i \quad \text{for} \quad v = \sum_{i=1}^n v_i X_i + \sum_{i=1}^n v_i' Y_i \in V_n.$$

For $v = \sum_{i=1}^n v_i X_i + v_i' Y_i \in V_n$ and $a \in \mathbb{R}$, we write

$$a \cdot v := \sum_{i=1}^n e^{a\lambda_i} v_i X_i + e^{a\lambda_i'} v_i' Y_i.$$

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The corresponding simply connected Lie group $G_{n,\mu}$, which is exponential solvable, can be identified with the space $\mathbb{R} \times V_n \times \mathbb{R}$ equipped with the multiplication

$$(a, v, c) \cdot (a', v', c') \tag{5}$$

$$= (a + a', (-a') \cdot v + v', e^{-2a'} c + c' + \frac{1}{2}\omega_n((-a') \cdot v, v')) \tag{6}$$

Denote by G_{V_n} the quotient group $G_{n,\mu}/\mathcal{Z}$ which can be identified with $\mathbb{R} \times V_n$ equipped with the multiplication

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We write $V_n = V_0 \oplus V_+ \oplus V_- = V_0 \oplus V_1$, where

$$\begin{array}{rcl} V_+ & := & \mathrm{span}\{X_j, Y_k; \lambda_j > 0, \lambda'_k > 0\}, \\ V_- & := & \mathrm{span}\{X_j; \lambda_j < 0\}, \\ V_0 & := & \mathrm{span}\{X_j, Y_k; \lambda_j = 0, \lambda'_k = 0\}, \end{array}$$

and $V_1 := V_+ \oplus V_-$. Let

$$\mu_+ := \mu \cap \mathbb{R}^*_+, \ \mu_- := \mu \cap \mathbb{R}^*_-, \ \mu_0 := \mu \cap \{0\},$$

then we can write

$$V_+ = \sum_{\lambda \in \mu_+} V_{+,\lambda} \quad \text{and} \quad V_- = \sum_{\lambda \in \mu_-} V_{-,\lambda},$$

where $V_{+,\lambda}$ and $V_{-,\lambda}$ are the respective eigenspaces of the operator l_μ.

We can also identify $\mathfrak{g}_{n,\mu}^*$ with $\mathbb{R}A^* \oplus V_n^* \oplus \mathbb{R}Z^* \simeq \mathbb{R} \times V_n \times \mathbb{R}$, and then

$$
\begin{aligned}
& \langle \operatorname{Ad}^*(a,u)(a^*,v^*,\lambda^*),(0,v,z)\rangle \\
= \;& \langle (a^*,v^*,\lambda^*),\ \operatorname{Ad}((a,u)^{-1})(0,v,z)\rangle \\
= \;& \langle (a^*,v^*,\lambda^*),(0,(-a)\cdot v, e^{-2a}z + e^{-2a}\omega_n(-(a\cdot u),v))\rangle \\
= \;& \langle 0,v^*,(-a)\cdot v\rangle + \lambda^* e^{-2a} z + \lambda^* e^{-2a}\omega_n(-(a\cdot u),v).
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Hence

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Here we denote by $u \times \omega_n$ the linear functional on V_n as

$$u \times \omega_n(v) := \omega_n(u,v) \quad \text{for all} \quad v \in V_n.$$

The coadjoint orbit Ω_ℓ of an element $\ell = (a^*, v^*, \lambda^*) \in \mathfrak{g}^*_{n,\mu}$ is given by

$$\Omega_\ell = \{(a^* + v^*([A, u]) + 2z\lambda^*, (-a) \cdot v^* - \lambda^* e^{-2a}(a \cdot u) \times \omega_n, \lambda^* e^{-2a})$$

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Hence if $\lambda^* \neq 0$ then the corresponding coadjoint orbit is the subset

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Therefore we have two open coadjoint orbits

$$\Omega_\varepsilon := \mathrm{Ad}^*(G_{n,\mu})\ell_\varepsilon = \mathbb{R} \times V_n^* \times \mathbb{R}^*_\varepsilon \quad \text{for} \quad \varepsilon \in \{+, -\}, \tag{7}$$

where $\ell_\varepsilon = \varepsilon Z^*$.

The other orbits are contained in $Z^{\perp}$ of the form

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or the one point orbits

$$\{a^* A^* + v^*\} \quad \text{for} \quad a^* \in \mathbb{R}, v^* \in V_0^*.$$

We can decompose the linear dual space V_n^* of V_n into

$$V_+^* := \{f \in V_n^* : f(V_- \cup V_0) = \{0\}\},$$
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Denote by $\|\cdot\|$ the norm on V_n^* coming from the scalar product defined by the basis $\{X_1, \cdots, X_n, Y_1, \cdots, Y_n\}$. For $f_+ = \sum_{\lambda \in \mu_+} f_\lambda \in V_+^*$ and $f_- = \sum_{\lambda \in \mu_-} f_\lambda \in V_-^*$, let

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$$|f_+|_\mu = |f_+| := \max_{\lambda_j\in\mu_+} \|f_j\|^{1/\lambda_j} \text{ and } |f_-|_\mu = |f_-| := \max_{\lambda_j\in\mu_-} \|f_j\|^{-1/\lambda_j}.$$

Then for $t \in \mathbb{R}$, we have the relation

$$|t \cdot f_+| = e^t |f_+| \text{ and } |t \cdot f_-| = e^{-t} |f_-| \text{ for } f_+ \in V_+^*, f_- \in V_-^*. \quad (8)$$

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On V_0^* we shall use the norm coming from the scalar product. This gives us a global gauge on V_n^*:

$$|(f_0, f_+, f_-)| := \max\{\|f_0\|, |f_+|, |f_-|\}.$$

We denote by V^*_{gen} the open subset of V^*_n consisting of all the $f = (f_0, f_+, f_-) \in V^*_0 \times V^*_+ \times V^*_-$ for which $f_+ \neq 0$ and $f_- \neq 0$. The subset V^*_{sin} consists of all the $f = (f_0, f_+, f_-)$ for which either $f_+ \neq 0, f_- = 0$ or $f_+ = 0, f_- \neq 0$.

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We see that for every $f = (f_0, f_+, f_-) \in V^*_{gen}$ there exists exactly one element $f' = (f_0, f'_+, f'_-)$ in its $G_{n,\mu}$-orbit such that $|f'_+| = |f'_-|$.

We denote by V_{gen}^* the open subset of V_n^* consisting of all the $f = (f_0, f_+, f_-) \in V_0^* \times V_+^* \times V_-^*$ for which $f_+ \neq 0$ and $f_- \neq 0$.
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We see that for every $f = (f_0, f_+, f_-) \in V_{gen}^*$ there exists exactly one element $f' = (f_0, f'_+, f'_-)$ in its $G_{n,\mu}$-orbit such that $|f'_+| = |f'_-|$.
In the same way, for $f = (f_0, f_+, 0)$ (resp. $f = (f_0, 0, f_-)) \in V_{sin}^*$, there exists exactly one element $f' = (f_0, f'_+, 0)$ (resp. $f' = (f_0, 0, f'_-)$) in its $G_{n,\mu}$-orbit for which $|f'_+| = 1$ (resp. $|f'_-| = 1$).

Let

$$\mathcal{D} = \{(f_0, f_+, f_-) : |f_+| = |f_-| \neq 0\},$$
$$\mathcal{S}_+ = \{(f_0, f_+, 0) : |f_+| = 1\}, \mathcal{S}_- = \{(f_0, 0, f_-) : |f_-| = 1\}, \text{ and}$$
$$\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-.$$

We assume for simplicity that $V^*_{gen} \neq \emptyset$, i.e., $\mu_+ \neq \emptyset$ and $\mu_- \neq \emptyset$.

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$$\begin{aligned}
\Gamma_0 &= \mathbb{R} \times V_0^*, \text{ corresponding to the unitary characters of } G_{n,\mu}, \\
\Gamma_1 &= \mathcal{S} \simeq V_{sin}^*/G_{n,\mu}, \\
\Gamma_2 &= \mathcal{D} \simeq V_{gen}^*/G_{n,\mu}, \\
\Gamma_3 &= \{+,-\} \simeq \{\Omega_+, \Omega_-\}/G_{n,\mu}.
\end{aligned}$$

Theorem

1. *A properly converging sequence* $(\Omega_{f_k})_k$ *with* $f_k = (f_{k,0}, f_{k_+}, f_{k_-}) \in \mathcal{D}$ *has either a unique limit point* Ω_f *for some* $f \in \mathcal{D}$ *and then* $f = \lim_k f_k$*, or* $\lim_k (f_{k_+}, f_{k_-}) = 0$ *and then the limit set* L *of the sequence is given by*

$$L = \{\Omega_{(f_0, f_+, 0)}, \Omega_{(f_0, 0, f_-)}, \mathbb{R}\},$$

   where $f_0 = \lim_k f_{k,0}$, $f_+ = \lim_k r(f_{k_+}) \cdot f_{k_+} \in \mathcal{S}_+$ *and* $f_- = \lim_k q(f_{k_-}) \cdot f_{k_-} \in \mathcal{S}_-$.

2. *A properly converging sequence* (Ω_{f_k}) *with* $f_k = (f_{k,0}, f_{k_+}, f_{k_-}) \in \mathcal{S}$ *has the limit set*

$$L = \{\Omega_f, \mathbb{R}\},$$

   where $f = \lim_k f_k \in \mathcal{S}$.

Corollary

The orbit Ω_f for $f \in \mathcal{D}$ is closed in $\mathfrak{g}^*_{n,\mu}$. The closure of the orbit Ω_f for $f \in \mathcal{S}$ is the set $\{\Omega_f, \mathbb{R}\}$.

Corollary

The orbit Ω_f *for* $f \in \mathcal{D}$ *is closed in* $\mathfrak{g}_{n,\mu}^*$. *The closure of the orbit* Ω_f *for* $f \in \mathcal{S}$ *is the set* $\{\Omega_f, \mathbb{R}\}$.

Corollary

For $\varepsilon \in \{+, -\}$, *the boundary of the open orbit* Ω_ε *is the subset* $\mathbb{R} \times V_n^* \times \{0\} = Z^\perp \simeq \mathfrak{g}_{V_n}^*$.

On the other hand, for every coadjoint orbit we can write down a corresponding irreducible representation as an induced representation by using Kirillov's orbit theory.

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Let $P = \exp(\sum_{j=1}^{n} \mathbb{R}Y_j + \mathbb{R}Z)$. This is a closed connected normal abelian subgroup of $G_{n,\mu}$. Let also $\mathfrak{x} := \sum_{j=1}^{n} \mathbb{R}X_j$ and $\mathfrak{y} := \sum_{j=1}^{n} \mathbb{R}Y_j \subset V_n$ (an abelian subalgebra of $\mathfrak{g}_{n,\mu}$), then $\mathcal{X} := \exp(\mathfrak{x})$ and $\mathcal{Y} = \exp(\mathfrak{y})$ are closed connected abelian subgroups of $G_{n,\mu}$. We have

$$G_{n,\mu} = \exp(\mathbb{R}A) \cdot \mathcal{X} \cdot P = S \cdot P,$$

where $S := \exp(\mathbb{R}A) \cdot \mathcal{X}$ is a subgroup of $G_{n,\mu}$. The irreducible representations $\pi_\varepsilon, \varepsilon = \pm$, corresponding to the orbits Ω_ε are of the form

$$\pi_\varepsilon := \mathrm{ind}_P^{G_{n,\mu}} \chi_{\varepsilon Z^*}.$$

The Hilbert space of π_ε is $L^2(S_n) = L^2(\mathbb{R}^{n+1})$.

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For $F \in L^1(G_{n,\mu})$ and $\xi \in L^2(G_{n,\mu}/P_n)$, the operator $\pi_\varepsilon(F)$ is given by the kernel function

$$\begin{aligned} & F_\varepsilon((a', x'), (a, x)) \\ = \; & \widehat{F}^{\mathfrak{p}_n}\Big(a' - a, a \cdot (x' - x); (-\varepsilon(\sum_{j=1}^{n} e^{(\lambda_j - 2)a} x_j Y_j^*), \varepsilon e^{-2a})\Big) e^{|\lambda| a}. \end{aligned}$$

For $v^* \in V_n^*$, the irreducible representation π_{v^*} on $L^2(\mathbb{R})$ is defined by

$$\pi_{v^*} := \mathrm{ind}_{H_n}^{G_{n,\mu}} \chi_{v^*},$$

where $H_n := \exp(\mathfrak{h}_n)$.

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The kernel function F_{v^*} of the operator $\pi_{v^*}(F), F \in L^1(G_{n,\mu})$, is given by

$$F_{v^*}(a, b) = \widehat{F}^{\mathfrak{h}_n}(a - b, b \cdot v^*, 0) \quad \text{for} \quad a, b \in \mathbb{R}. \tag{9}$$

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Finally, for $(a^*, v_0^*) \in \mathbb{R} \times V_0^*$ we have the unitary characters

$$\chi_{(a^*, v_0^*)(a, v_0, v, c)} := e^{-2\pi i (a^* a + v_0^*(v_0))} \text{ for } a, c \in \mathbb{R}, v_0 \in V_0, v \in V_1.$$

Almost $C_0(\mathcal{K})$-C*-algebras

Definition

Let A be a C*-algebra and $\widehat{A}$ be the spectrum of A.

1. Suppose that the spectrum $\hat{A} = \cup_{i=0}^{d} \Gamma_i$ is Hausdorff of step d. Furthermore we assume that for every $i \in \{0, \cdots, d\}$ there exists a Hilbert space $\mathcal{H}_i$ and that a concrete realization $(\pi_\gamma, \mathcal{H}_i)$ of γ on the Hilbert space $\mathcal{H}_i$ for every $\gamma \in \Gamma_i$ is chosen.

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   The set S_0 is the collection $\mathcal{X}$ of all characters of A.
2. For a subset $S \subset \widehat{A}$, denote by $CB(S)$ the *-algebra of all uniformly bounded operator fields $(\psi(\gamma) \in \mathcal{B}(\mathcal{H}_i))_{\gamma \in S \cap \Gamma_i, i=1,\cdots,d}$, which are operator norm continuous on the subsets $\Gamma_i \cap S$ for every $i \in \{1, \cdots, d\}$ for which $\Gamma_i \cap S \neq \emptyset$. We provide the *-algebra $CB(S)$ with the infinity-norm:

$$\|\psi\|_S := \sup_{\gamma \in S} \|\psi(\gamma)\|_{\mathrm{op}}.$$

Definition

A C*-algebra A is said to be *almost* $C_0(\mathcal{K})$ if for every $a \in A$:

1. The mappings $\gamma \mapsto \mathcal{F}(a)(\gamma)$ are norm continuous on the different sets Γ_i, where $\mathcal{F} : A \to l^\infty(\widehat{A})$ is the Fourier transform given by

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2. For each $i = 1, \cdots, d$, we have a sequence $(\sigma_{i,k} : CB(S_{i-1}) \to CB(S_i))_k$ of linear mappings which are uniformly bounded in k (and independent of a) such that

$$\lim_{k\to\infty} \operatorname{dis}\Big((\sigma_{i,k}(\mathcal{F}(a)_{|S_{i-1}}) - \mathcal{F}(a)_{|\Gamma_i}), C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i)) \Big) = 0,$$

and

$$\lim_{k\to\infty} \operatorname{dis}\Big((\sigma_{i,k}(\mathcal{F}(a)^*_{|S_{i-1}}) - \mathcal{F}(a^*)_{|\Gamma_i}), C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i)) \Big) = 0.$$

Definition

Let $D^*(A)$ be the set of all operator fields φ defined over $\widehat{A}$ such that

1. The field φ is uniformly bounded, i.e., we have that $\|\varphi\| := \sup_{\gamma \in \widehat{A}} \|\varphi(\gamma)\|_{\text{op}} < \infty$.
2. $\varphi_{|\Gamma_i} \in CB(\Gamma_i)$ for every $i = 0, 1, \ldots, d$.

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Definition

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4. For each $i = 1, 2, \ldots, d$,

$$\lim_{k \to \infty} \mathrm{dis}\Big((\sigma_{i,k}(\varphi_{|S_{i-1}}) - \varphi_{|\Gamma_i}), C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i))\Big) = 0$$

and

$$\lim_{k \to \infty} \mathrm{dis}\Big((\sigma_{i,k}(\varphi^*_{|S_{i-1}}) - (\varphi_{|\Gamma_i})^*), C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i))\Big) = 0.$$

We see immediately that if A is almost $C_0(\mathcal{K})$, then for every $a \in A$, the operator field $\mathcal{F}(a)$ is contained in the set $D^*(A)$. In fact it turns out that $D^*(A)$ is a C*-subalgebra of $l^\infty(\widehat{A})$ and that A is isomorphic to $D^*(A)$.

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Theorem

(Inoue-Lin-L.) Let A be a separable C-algebra which is almost $C_0(\mathcal{K})$. Then the subset $D^*(A)$ of the C*-algebra $l^\infty(\widehat{A})$ is a C*-subalgebra which is isomorphic to A under the Fourier transform.*

Theorem

(Inoue-Lin-L.) The C-algebra of* $G_{n,\mu}$ *is an almost* $C_0(\mathcal{K})$*-C*-algebra. In particular, the Fourier transform maps* $C^*(G_{n,\mu})$ *onto the subalgebra* $D^*(G_{n,\mu})$ *of* $l^\infty(\widehat{G_{n,\mu}})$*.*

Thank you