Multipliers of dynamical systems

Andrew McKee
with I. Todorov and L. Turowska

Queen's University Belfast

August 2016

Background and motivation

Background and motivation

(X, μ), (Y, ν) standard measure spaces.

Background and motivation

(X, μ), (Y, ν) standard measure spaces. $k \in L^2(Y \times X)$ defines **Hilbert–Schmidt operator** $T_k \in \mathcal{B}(L^2(X), L^2(Y))$ by

$$(T_k \xi)(y) := \int_X k(y, x)\xi(x)\, d\mu(x).$$

Background and motivation

(X,μ), (Y,ν) standard measure spaces. $k \in L^2(Y \times X)$ defines **Hilbert–Schmidt operator** $T_k \in \mathcal{B}(L^2(X), L^2(Y))$ by

$$(T_k\xi)(y) := \int_X k(y,x)\xi(x)\,d\mu(x).$$

Given $\varphi \in L^\infty(X \times Y)$ define $\varphi k \in L^2(Y \times X)$ by

$$\varphi k(y,x) := \varphi(x,y)k(y,x).$$

Background and motivation

(X,μ), (Y,ν) standard measure spaces. $k \in L^2(Y \times X)$ defines **Hilbert–Schmidt operator** $T_k \in \mathcal{B}(L^2(X), L^2(Y))$ by

$$(T_k\xi)(y) := \int_X k(y,x)\xi(x)\, d\mu(x).$$

Given $\varphi \in L^\infty(X \times Y)$ define $\varphi k \in L^2(Y \times X)$ by

$$\varphi k(y,x) := \varphi(x,y)k(y,x).$$

Definition

$\varphi \in L^\infty(X \times Y)$ *is called a Schur multiplier if the map*

$$S_\varphi : T_k \mapsto T_{\varphi k}$$

is **bounded** *in the operator norm.*

Background and motivation

(X, μ), (Y, ν) standard measure spaces. $k \in L^2(Y \times X)$ defines **Hilbert–Schmidt operator** $T_k \in \mathcal{B}(L^2(X), L^2(Y))$ by

$$(T_k \xi)(y) := \int_X k(y, x)\xi(x)\, d\mu(x).$$

Given $\varphi \in L^\infty(X \times Y)$ define $\varphi k \in L^2(Y \times X)$ by

$$\varphi k(y, x) := \varphi(x, y) k(y, x).$$

Definition

$\varphi \in L^\infty(X \times Y)$ is called a Schur multiplier if the map

$$S_\varphi : T_k \mapsto T_{\varphi k}$$

is **bounded** *in the operator norm.*

It turns out that S_φ is **completely bounded**.

Theorem (Haagerup, Peller)

Let $\varphi \in L^\infty(X \times Y)$. *TFAE:*

i. φ *is a Schur multiplier;*

ii. *there exist families* $(a_k)_{\mathbb{N}}$ *and* $(b_k)_{\mathbb{N}}$ *of measurable functions with* $\mathrm{esssup}_x \sum_k |a_k(x)|^2 < \infty,\ \mathrm{esssup}_y \sum_k |b_k(y)|^2 < \infty$ *such that*

$$\varphi(x, y) = \sum_{k=1}^{\infty} a_k(x) b_k(y).$$

G a **locally-compact group**.

Definition

$u : G \to \mathbb{C}$ *is a Herz–Schur multiplier if it is a* **completely bounded multiplier** *of* $A(G)$,

G a **locally-compact group**.

Definition

$u : G \to \mathbb{C}$ is a Herz–Schur multiplier if it is a **completely bounded multiplier** *of $A(G)$, i.e.*

$$m_u : A(G) \to A(G);\ v \mapsto uv$$

is completely bounded.

G a **locally-compact group**.

Definition

$u : G \to \mathbb{C}$ is a *Herz–Schur multiplier* *if it is a* **completely bounded multiplier** *of $A(G)$, i.e.*

$$m_u : A(G) \to A(G);\ v \mapsto uv$$

is completely bounded.

Given $u : G \to \mathbb{C}$ define $N(u) : G \times G \to \mathbb{C}$ by

$$N(u)(s,t) := u(ts^{-1}).$$

G a **locally-compact group**.

Definition

$u : G \to \mathbb{C}$ is a Herz–Schur multiplier if it is a **completely bounded multiplier** *of $A(G)$, i.e.*

$$m_u : A(G) \to A(G); \; v \mapsto uv$$

is completely bounded.

Given $u : G \to \mathbb{C}$ define $N(u) : G \times G \to \mathbb{C}$ by

$$N(u)(s,t) := u(ts^{-1}).$$

Theorem (Bożejko–Fendler)

Let $u : G \to \mathbb{C}$. TFAE:

- i. *u is a Herz–Schur multiplier;*
- ii. *$N(u)$ is a Schur multiplier on $G \times G$.*

Why are we interested?

Why are we interested?

- generalisation of entrywise product of matrices;

Why are we interested?

- generalisation of entrywise product of matrices;
- $M^{\mathrm{cb}}A(G)$;

Why are we interested?

- generalisation of entrywise product of matrices;
- $M^{\mathrm{cb}}A(G)$;
- Herz–Schur multipliers give rise to completely bounded maps on $\mathrm{vN}(G)$ and $C_r^*(G)$, so link properties of a group and its associated operator algebras...

Why are we interested?

- generalisation of entrywise product of matrices;
- $M^{\mathrm{cb}}A(G)$;
- Herz–Schur multipliers give rise to completely bounded maps on $\mathrm{vN}(G)$ and $C_r^*(G)$, so link properties of a group and its associated operator algebras...
- ...approximation properties.

Framework

Let $(X, \mu), (Y, \nu)$ **standard measure spaces**, $A \subseteq \mathcal{B}(\mathcal{H})$ a C^*-**algebra**.

Framework

Let $(X, \mu), (Y, \nu)$ **standard measure spaces**, $A \subseteq \mathcal{B}(\mathcal{H})$ a C^*-**algebra**.
Given $k \in L^2(Y \times X, A)$ define $T_k \in \mathcal{B}(L^2(X, \mathcal{H}), L^2(Y, \mathcal{H}))$

$$(T_k \xi)(y) := \int_X k(y, x)\big(\xi(x)\big)\, d\mu(x), \quad \xi \in L^2(X, \mathcal{H}).$$

Then $\|T_k\| \leq \|k\|_2$ and $k = 0 \;\Leftrightarrow T_k = 0$.

Framework

Let $(X,\mu),(Y,\nu)$ **standard measure spaces**, $A\subseteq\mathcal{B}(\mathcal{H})$ a C^*-**algebra**.
Given $k\in L^2(Y\times X,A)$ define $T_k\in\mathcal{B}(L^2(X,\mathcal{H}),L^2(Y,\mathcal{H}))$

$$(T_k\xi)(y):=\int_X k(y,x)\big(\xi(x)\big)\,d\mu(x),\quad \xi\in L^2(X,\mathcal{H}).$$

Then $\|T_k\|\leq\|k\|_2$ and $k=0 \;\Leftrightarrow\; T_k=0$.
Let $\mathcal{S}_2(Y\times X,A)$ denote the space of all T_k; operator space structure as a dense subspace of $\mathcal{K}(L^2(X),L^2(Y))\otimes A$.

Framework

Let $(X,\mu),(Y,\nu)$ **standard measure spaces**, $A \subseteq \mathcal{B}(\mathcal{H})$ a C^*-**algebra**.
Given $k \in L^2(Y \times X, A)$ define $T_k \in \mathcal{B}(L^2(X,\mathcal{H}), L^2(Y,\mathcal{H}))$

$$(T_k\xi)(y) := \int_X k(y,x)\big(\xi(x)\big)\, d\mu(x), \quad \xi \in L^2(X,\mathcal{H}).$$

Then $\|T_k\| \leq \|k\|_2$ and $k = 0 \iff T_k = 0$.
Let $\mathcal{S}_2(Y \times X, A)$ denote the space of all T_k; operator space structure as a dense subspace of $\mathcal{K}(L^2(X), L^2(Y)) \otimes A$.
Given $\varphi : X \times Y \to \mathcal{CB}(A, \mathcal{B}(\mathcal{H}))$ bounded, pointwise-measurable define

$$\varphi \cdot k(y,x) := \varphi(x,y)\big(k(y,x)\big).$$

Then $\varphi \cdot k \in L^2(Y \times X, A)$ and $\|\varphi \cdot k\|_2 \leq \|\varphi\|_\infty \|k\|_2$.

Schur A-multipliers

Definition
$\varphi : X \times Y \to \mathcal{CB}(A, \mathcal{B}(\mathcal{H}))$ *is called a Schur A-multiplier if*

$$S_\varphi : T_k \mapsto T_{\varphi \cdot k}$$

is **completely bounded**. $\|\varphi\|_S := \|S_\varphi\|_{cb}$.

Schur A-multipliers

Definition

$\varphi : X \times Y \to \mathcal{CB}(A, \mathcal{B}(\mathcal{H}))$ is called a Schur A-multiplier if

$$S_\varphi : T_k \mapsto T_{\varphi \cdot k}$$

is **completely bounded**. $\|\varphi\|_S := \|S_\varphi\|_{cb}$.

It is convenient to work with Schur A-multipliers of the form $\varphi : X \times Y \to \mathcal{CB}(A)$ as they are independent of the faithful, separable representation of A.

Theorem

Let $\varphi : X \times Y \to \mathcal{CB}(A)$ *be a bounded pointwise-measurable function. The following are equivalent:*

i. φ *is a Schur A-multiplier;*

Theorem

Let $\varphi : X \times Y \to \mathcal{CB}(A)$ be a bounded pointwise-measurable function. The following are equivalent:

i. *φ is a Schur A-multiplier;*

ii. *there exist a separable Hilbert space $\mathcal{K}$, a non-degenerate *-representation $\rho : A \to \mathcal{B}(\mathcal{K})$, $V \in L^\infty(X, \mathcal{B}(\mathcal{H}, \mathcal{K}))$, and $W \in L^\infty(Y, \mathcal{B}(\mathcal{H}, \mathcal{K}))$, such that*

$$\varphi(x, y)(a) = W^*(y)\rho(a)V(x), \quad a \in A,$$

for almost all $(x, y) \in X \times Y$.

More framework

G locally compact group (left Haar measure), $\alpha : G \to \mathrm{Aut}(A)$ point-norm continuous homomorphism, i.e. (A, G, α) a **C*-dynamical system**.

More framework

G locally compact group (left Haar measure), $\alpha : G \to \mathrm{Aut}(A)$ point-norm continuous homomorphism, i.e. (A, G, α) a **C*-dynamical system**. For $\xi \in L^2(G, \mathcal{H})$

$$(\pi(a)\xi)(t) := \alpha_{t^{-1}}(a)\big(\xi(t)\big), \quad (\lambda_s\xi)(t) := \xi(s^{-1}t)$$

defines a covariant representation (π, λ) of (A, G, α) on $L^2(G, \mathcal{H})$,

More framework

G locally compact group (left Haar measure), $\alpha : G \to \mathrm{Aut}(A)$ point-norm continuous homomorphism, i.e. (A, G, α) a **C*-dynamical system**. For $\xi \in L^2(G, \mathcal{H})$

$$(\pi(a)\xi)(t) := \alpha_{t^{-1}}(a)\big(\xi(t)\big), \quad (\lambda_s\xi)(t) := \xi(s^{-1}t)$$

defines a covariant representation (π, λ) of (A, G, α) on $L^2(G, \mathcal{H})$, i.e. $\pi(\alpha_t(a)) = \lambda_t \pi(a) \lambda_t^*$.

More framework

G locally compact group (left Haar measure), $\alpha : G \to \mathrm{Aut}(A)$ point-norm continuous homomorphism, i.e. (A, G, α) a **C*-dynamical system**. For $\xi \in L^2(G, \mathcal{H})$

$$(\pi(a)\xi)(t) := \alpha_{t^{-1}}(a)\big(\xi(t)\big), \quad (\lambda_s\xi)(t) := \xi(s^{-1}t)$$

defines a covariant representation (π, λ) of (A, G, α) on $L^2(G, \mathcal{H})$, i.e. $\pi(\alpha_t(a)) = \lambda_t\pi(a)\lambda_t^*$. We get a representation of $L^1(G, A)$ by

$$f \mapsto \pi \rtimes \lambda(f) := \int_G \pi\big(f(s)\big)\lambda_s\, ds, \quad f \in L^1(G, A).$$

Completion of $L^1(G, A)$ in associated **operator norm** is reduced crossed product $A \rtimes_{\alpha,r} G$.

Herz–Schur (A, G, α)-multipliers

Let $f \in L^1(G, A)$ and $F : G \to \mathcal{CB}(A)$ weakly measurable; define $F \cdot f \in L^1(G, A)$

$$F \cdot f(t) := F(t)\big(f(t)\big), t \in G.$$

$\|F \cdot f\|_1 \leq \|F\|_\infty \|f\|_1$

Definition

$F : G \to \mathcal{CB}(A)$ is a Herz–Schur (A, G, α)-multiplier if

$$S_F : \pi \rtimes \lambda(f) \mapsto \pi \rtimes \lambda(F \cdot f)$$

is **completely bounded**. *S_F extends to a completely bounded map on $A \rtimes_{\alpha,r} G$.*

Herz–Schur (A, G, α)-multipliers

Let $f \in L^1(G, A)$ and $F : G \to \mathcal{CB}(A)$ weakly measurable; define $F \cdot f \in L^1(G, A)$

$$F \cdot f(t) := F(t)\big(f(t)\big), t \in G.$$

$\|F \cdot f\|_1 \leq \|F\|_\infty \|f\|_1$

Definition

$F : G \to \mathcal{CB}(A)$ is a Herz–Schur (A, G, α)-multiplier if

$$S_F : \pi \rtimes \lambda(f) \mapsto \pi \rtimes \lambda(F \cdot f)$$

is **completely bounded***. S_F extends to a completely bounded map on $A \rtimes_{\alpha,r} G$.*
If S_F is **bounded** *(not cb) we speak of a multiplier of (A, G, α).*

These properties are independent of faithful representation of A.

Let Γ be a locally compact group, X a space. For $\phi : G \to X$ let

$$\phi^{\Gamma} : \Gamma \times G \to X; \; \phi^{\Gamma}(\gamma, t) := \phi(t).$$

Let Γ be a locally compact group, X a space. For $\phi : G \to X$ let

$$\phi^\Gamma : \Gamma \times G \to X; \ \phi^\Gamma(\gamma, t) := \phi(t).$$

The following extends a classical characterisation of de Cannière–Haagerup.
We say a multiplier of (A, G, α) is weak*-extendible if S_F extends to a weak*-continuous map on $\overline{A \rtimes_{\alpha,r} G}^{w^*}$.

Proposition

Let $F : G \to \mathcal{CB}(A)$ be a bounded, pointwise measurable function. TFAE:

i. *F is a weak*-extendible Herz–Schur (A, G, α)-multiplier;*

Let Γ be a locally compact group, X a space. For $\phi : G \to X$ let

$$\phi^\Gamma : \Gamma \times G \to X; \ \phi^\Gamma(\gamma, t) := \phi(t).$$

The following extends a classical characterisation of de Cannière–Haagerup.
We say a multiplier of (A, G, α) is weak*-extendible if S_F extends to a weak*-continuous map on $\overline{A \rtimes_{\alpha,r} G}^{w^*}$.

Proposition

Let $F : G \to \mathcal{CB}(A)$ be a bounded, pointwise measurable function. TFAE:

i. *F is a weak*-extendible Herz–Schur (A, G, α)-multiplier;*
ii. *for any Γ, F^Γ is a weak*-extendible multiplier of $(A, \Gamma \times G, \alpha^\Gamma)$;*
iii. *$F^{SU(2)}$ is a weak*-extendible multiplier of $(A, SU(2) \times G, \alpha^{SU(2)})$.*

Proposition

Let $u : G \to \mathbb{C}$ *be bounded, continuous;* $F_u(t)(a) := u(t)a$*.*

Proposition

Let $u : G \to \mathbb{C}$ *be bounded, continuous;* $F_u(t)(a) := u(t)a$. *TFAE:*

i. F_u *is a Herz–Schur* (A, G, α)*-multiplier;*

ii. u *is a classical Herz–Schur multiplier.*

Transference

Transference

Let $F : G \to \mathcal{CB}(A)$ be pointwise-measurable, define $\mathcal{N}(F) : G \times G \to \mathcal{CB}(A)$

$$\mathcal{N}(F)(s,t)(a) := \alpha_{t^{-1}}\big(F(ts^{-1})(\alpha_t(a))\big).$$

Transference

Let $F : G \to \mathcal{CB}(A)$ be pointwise-measurable, define $\mathcal{N}(F) : G \times G \to \mathcal{CB}(A)$

$$\mathcal{N}(F)(s,t)(a) := \alpha_{t^{-1}}\big(F(ts^{-1})(\alpha_t(a))\big).$$

Theorem

Let $F : G \to \mathcal{CB}(A)$ be bounded, pointwise-measurable. TFAE:

i. *F is a Herz–Schur (A, G, α)-multiplier;*

ii. *$\mathcal{N}(F)$ is a Schur A-multiplier.*

Let ρ be the right regular representation of G on $L^2(G)$ given by

$$(\rho_s \xi)(t) := \Delta(s)^{-1} \xi(ts).$$

Let ρ be the right regular representation of G on $L^2(G)$ given by

$$(\rho_s \xi)(t) := \Delta(s)^{-1} \xi(ts).$$

For each $t \in G$ let

$$\tilde{\alpha}_t := (\mathrm{Ad}\, \rho_t) \otimes \alpha_t.$$

Definition

A Schur A-multiplier $\varphi : G \times G \to \mathcal{CB}(A)$ will be called invariant if S_φ commutes with $\tilde{\alpha}_t$ for every $t \in G$.

Let ρ be the right regular representation of G on $L^2(G)$ given by

$$(\rho_s \xi)(t) := \Delta(s)^{-1}\xi(ts).$$

For each $t \in G$ let

$$\tilde{\alpha}_t := (\mathrm{Ad}\,\rho_t) \otimes \alpha_t.$$

Definition

A Schur A-multiplier $\varphi : G \times G \to \mathcal{CB}(A)$ will be called invariant if S_φ commutes with $\tilde{\alpha}_t$ for every $t \in G$.

Theorem

$\mathcal{N}$ is a linear isometry of the Herz–Schur (A, G, α)-multipliers onto the **invariant** *Schur A-multipliers on $G \times G$.*

Thank you!