Multipliers of dynamical systems

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Background and motivation

Given $\phi \in L^\infty(X \times Y)$ define $\phi_k \in L^2(Y \times X)$ by $\phi_k(y, x) := \phi(x, y) k(y, x)$.

Definition $\phi \in L^\infty(X \times Y)$ is called a Schur multiplier if the map $S\phi : k \mapsto \phi_k$ is bounded in the operator norm.

It turns out that $S\phi$ is completely bounded.
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$$(T_k \xi)(y) := \int_X k(y, x) \xi(x) \, d\mu(x).$$
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is bounded in the operator norm.

It turns out that \(S_\varphi\) is completely bounded.
Theorem (Haagerup, Peller)

Let \( \varphi \in L^\infty(X \times Y) \). TFAE:

i. \( \varphi \) is a Schur multiplier;

ii. there exist families \((a_k)_N\) and \((b_k)_N\) of measurable functions with \( \text{esssup}_x \sum_k |a_k(x)|^2 < \infty \), \( \text{esssup}_y \sum_k |b_k(y)|^2 < \infty \) such that

\[
\varphi(x, y) = \sum_{k=1}^{\infty} a_k(x) b_k(y).
\]
$G$ a locally-compact group.

**Definition**

$u : G \rightarrow \mathbb{C}$ is a *Herz–Schur multiplier* if it is a **completely bounded multiplier** of $\mathcal{A}(G)$.

Theorem (Bożejko–Fendler)

Let $u : G \rightarrow \mathbb{C}$. TFAE:

i. $u$ is a Herz–Schur multiplier;

ii. $N(u)$ is a Schur multiplier on $G \times G$. 
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$u : G \to \mathbb{C}$ is a *Herz–Schur multiplier* if it is a **completely bounded multiplier** of $A(G)$, i.e.

$$m_u : A(G) \to A(G); \ v \mapsto uv$$

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- generalisation of entrywise product of matrices;
- $M^{\text{cb}} A(G)$;
- Herz–Schur multipliers give rise to completely bounded maps on $\text{vN}(G)$ and $C_r^*(G)$, so link properties of a group and its associated operator algebras...
- ...approximation properties.
Framework

Let \((X, \mu), (Y, \nu)\) standard measure spaces, \(A \subseteq \mathcal{B}(\mathcal{H})\) a \(C^*\)-algebra.
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Given \(k \in L^2(Y \times X, A)\) define \(T_k \in \mathcal{B}(L^2(X, \mathcal{H}), L^2(Y, \mathcal{H}))\)

\[
(T_k \xi)(y) := \int_X k(y, x)(\xi(x)) \, d\mu(x), \quad \xi \in L^2(X, \mathcal{H}).
\]

Then \(\|T_k\| \leq \|k\|_2\) and \(k = 0 \iff T_k = 0\).
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$$(T_k \xi)(y) := \int_X k(y, x)(\xi(x)) \, d\mu(x), \quad \xi \in L^2(X, \mathcal{H}).$$

Then $\|T_k\| \leq \|k\|_2$ and $k = 0 \iff T_k = 0$. Let $S_2(Y \times X, A)$ denote the space of all $T_k$; operator space structure as a dense subspace of $\mathcal{K}(L^2(X), L^2(Y)) \otimes A$. 
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Let \(S_2(Y \times X, A)\) denote the space of all \(T_k\); operator space structure as a dense subspace of \(\mathcal{K}(L^2(X), L^2(Y)) \otimes A\).

Given \(\varphi : X \times Y \to CB(A, \mathcal{B}(\mathcal{H}))\) bounded, pointwise-measurable define

\[ \varphi \cdot k(y, x) := \varphi(x, y)(k(y, x)). \]

Then \(\varphi \cdot k \in L^2(Y \times X, A)\) and \(\|\varphi \cdot k\|_2 \leq \|\varphi\|_{\infty} \|k\|_2\).
Definition

\( \varphi : X \times Y \to \mathcal{CB}(A, \mathcal{B}(\mathcal{H})) \) is called a \textit{Schur A-multiplier} if

\[
S_\varphi : T_k \mapsto T_{\varphi \cdot k}
\]

is completely bounded. \( \| \varphi \|_S := \| S_\varphi \|_{cb} \).
Schur $A$-multipliers

Definition

$\varphi : X \times Y \rightarrow CB(A, B(\mathcal{H}))$ is called a Schur $A$-multiplier if

$$S_{\varphi} : T_k \mapsto T_{\varphi \cdot k}$$

is completely bounded. $\|\varphi\|_{S} := \|S_{\varphi}\|_{cb}$.

It is convenient to work with Schur $A$-multipliers of the form $\varphi : X \times Y \rightarrow CB(A)$ as they are independent of the faithful, separable representation of $A$. 
Theorem
Let $\varphi : X \times Y \rightarrow CB(A)$ be a bounded pointwise-measurable function. The following are equivalent:

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Let $\varphi : X \times Y \to CB(A)$ be a bounded pointwise-measurable function. The following are equivalent:

i. \( \varphi \) is a Schur A-multiplier;

ii. there exist a separable Hilbert space $\mathcal{K}$, a non-degenerate $\ast$-representation $\rho : A \to \mathcal{B}(\mathcal{K})$, $V \in L^\infty(X, \mathcal{B}(\mathcal{H}, \mathcal{K}))$, and $W \in L^\infty(Y, \mathcal{B}(\mathcal{H}, \mathcal{K}))$, such that

$$
\varphi(x, y)(a) = W^*(y)\rho(a)V(x), \quad a \in A,
$$

for almost all $(x, y) \in X \times Y$. 
More framework

$G$ locally compact group (left Haar measure), $\alpha : G \to \text{Aut}(A)$ point-norm continuous homomorphism, i.e. $(A, G, \alpha)$ a $\mathbb{C}^*$-dynamical system.
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$$(\pi(a)\xi)(t) := \alpha_{t^{-1}}(a)(\xi(t)), \quad (\lambda_s\xi)(t) := \xi(s^{-1}t)$$

defines a covariant representation $(\pi, \lambda)$ of $(A, G, \alpha)$ on $L^2(G, \mathcal{H})$, 
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defines a covariant representation $(\pi, \lambda)$ of $(A, G, \alpha)$ on $L^2(G, \mathcal{H})$, i.e. $\pi(\alpha_t(a)) = \lambda_t \pi(a) \lambda_t^*$. We get a representation of $L^1(G, A)$ by

$$f \mapsto \pi \ltimes \lambda(f) := \int_G \pi(f(s)) \lambda_s \, ds, \quad f \in L^1(G, A).$$

Completion of $L^1(G, A)$ in associated operator norm is reduced crossed product $A \rtimes_{\alpha,r} G$. 
**Herz–Schur \((A, G, \alpha)\)-multipliers**

Let \(f \in L^1(G, A)\) and \(F : G \to CB(A)\) weakly measurable; define \(F \cdot f \in L^1(G, A)\)

\[
F \cdot f(t) := F(t)(f(t)), \; t \in G.
\]

\[
\|F \cdot f\|_1 \leq \|F\|_\infty \|f\|_1
\]

**Definition**

\(F : G \to CB(A)\) is a **Herz–Schur \((A, G, \alpha)\)-multiplier** if

\[
S_F : \pi \rtimes \lambda(f) \mapsto \pi \rtimes \lambda(F \cdot f)
\]

is completely bounded. \(S_F\) extends to a completely bounded map on \(A \rtimes_{\alpha,r} G\).
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is **completely bounded**. \(S_F\) extends to a completely bounded map on \(A \rtimes_{\alpha, r} G\).

If \(S_F\) is **bounded** (not cb) we speak of a **multiplier of \((A, G, \alpha)\)**.

These properties are independent of faithful representation of \(A\).
Let $\Gamma$ be a locally compact group, $X$ a space. For $\phi : G \to X$ let

$$\phi^\Gamma : \Gamma \times G \to X; \phi^\Gamma(\gamma, t) := \phi(t).$$
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The following extends a classical characterisation of de Cannière–Haagerup.

We say a multiplier of $(A, G, \alpha)$ is \textit{weak*-extendible} if $S_F$ extends to a weak*-continuous map on $\overline{A \rtimes_\alpha r G}^{w*}$.

**Proposition**

Let $F : G \to CB(A)$ be a bounded, pointwise measurable function. TFAE:

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**Proposition**

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ii. for any $\Gamma$, $F^\Gamma$ is a weak*-extendible multiplier of $(A, \Gamma \times G, \alpha^\Gamma)$;

iii. $F^{SU(2)}$ is a weak*-extendible multiplier of $(A, SU(2) \times G, \alpha^{SU(2)})$. 
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i. $F_u$ is a Herz–Schur $(A, G, \alpha)$-multiplier;

ii. $u$ is a classical Herz–Schur multiplier.
Transference

Let $F : G \to CB(A)$ be pointwise-measurable, define $N(F) : G \times G \to CB(A)$ by

$$N(F)(s, t)(a) := \alpha t^{-1}(F(ts^{-1})(\alpha t(a))).$$

**Theorem**

Let $F : G \to CB(A)$ be bounded, pointwise-measurable. TFAE:

i. $F$ is a Herz–Schur $(A, G, \alpha)$-multiplier;

ii. $N(F)$ is a Schur $A$-multiplier.
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For each $t \in G$ let

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**Definition**

A Schur $A$-multiplier $\varphi : G \times G \to \mathcal{CB}(A)$ will be called invariant if $S_\varphi$ commutes with $\tilde{\alpha}_t$ for every $t \in G$. 

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**Theorem**

$\mathcal{N}$ is a linear isometry of the Herz–Schur $(A, G, \alpha)$-multipliers onto the **invariant** Schur $A$-multipliers on $G \times G$.
Thank you!