Non-separability of the Gelfand space of measure algebras

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Definitions

$M(\mathbb{T})$ – the Banach algebra of complex-valued, Borel regular measures on the circle group with the usual convolution as multiplication.
For $\mu \in M(\mathbb{T})$ we define the spectrum $\sigma(\mu)$ of μ as the set

$$\sigma(\mu) = \{\lambda \in \mathbb{C} : \mu - \lambda\delta_0 \text{ is not invertible}\}.$$

It follows from the general theory of commutative Banach algebras that the spectrum of a measure is an image of its Gelfand transform and it is non-empty compact subset of the complex plane. The alternative definition of the spectrum goes as follows: with every $\mu \in M(\mathbb{T})$ we can associate an operator $T_\mu : L^1(\mathbb{T}) \mapsto L^1(\mathbb{T})$ by the formula $T_\mu(f) = \mu * f$ and then $\sigma(\mu) = \sigma(T_\mu)$.
For $\mu \in M(\mathbb{T})$ we also define the n-th Fourier-Stieltjes coefficient:

$$\widehat{\mu}(n) = \int_{\mathbb{T}} e^{-int} d\mu(t).$$

One can also easily check that $\widehat{\mu}(\mathbb{Z}) = \sigma_p(T_\mu)$.

Formulation of the problem

Main problem

It is obvious that for every $\mu \in M(\mathbb{T})$ we have $\overline{\widehat{\mu}(\mathbb{Z})} \subset \sigma(\mu)$. But when do we have $\overline{\widehat{\mu}(\mathbb{Z})} = \sigma(\mu)$?

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Natural spectrum

Following M. Zafran we say that a measure $\mu \in M(\mathbb{T})$ has a natural spectrum iff $\overline{\widehat{\mu}(\mathbb{Z})} = \sigma(\mu)$.

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Examples

Absolutely continuous and purely discrete measures have a natural spectrum.

The Wiener-Pitt phenomenon

There exists a measure $\mu \in M(\mathbb{T})$ for which $\overline{\widehat{\mu}(\mathbb{Z})} \neq \sigma(\mu)$.

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The Proof of the existence of the Wiener-Pitt phenomenon by C. C. Graham

Consider the usual Riesz product:

$$\mu = \prod_{k=1}^{\infty}(1 + \cos(3^k t)) \text{ understood as a weak}^* \text{ limit of finite products.}$$

Then

$$\overline{\widehat{\mu}(\mathbb{Z})} = \{0\} \cup \left\{\frac{1}{2^n}\right\}_{n=1}^{\infty} \cup \{1\}.$$

The proof continued

The formula for the Fourier-Stieltjes coefficients of the Riesz product is

$$\widehat{\mu}\left(\sum_{k=1}^{n} \varepsilon_k 3^k\right) = \prod_{k=1}^{n}\left(\frac{1}{2}\right)^{|\varepsilon_k|} \text{ where } \varepsilon_k \in \{-1,0,1\} \text{ and}$$

$\widehat{\mu}(m) = 0$ if m is not expressible in the above form.
Suppose that $\sigma(\mu) = \overline{\widehat{\mu}(\mathbb{Z})}$ and let us take two disjoint open sets $A, B \subset \mathbb{C}$ such that $\overline{\widehat{\mu}(\mathbb{Z})} \subset A \cup B$ and $\sigma(\mu) \cap B = \left\{\frac{1}{2}\right\}$. Then the function f defined on $A \cup B$ by the formula $f = 1$ on B and $f = 0$ on A is holomorphic on $A \cup B$ and hence we can apply functional calculus obtaining a measure $\nu := f(\mu)$. By the properties of the functional calculus we have

$$\widehat{\nu}(\pm 3^k) = 1 \text{ and } \widehat{\nu}(n) = 0 \text{ for } n \text{ not of the form } \pm 3^k.$$

This result contradicts the classical Helson's theorem (finitely-valued Fourier-Stieltjes sequence has to be periodic with only finitely many exceptions).

The proof of the non-separability of $\triangle(M(\mathbb{T}))$ I

We shall need the following simple arithmetic lemma which follows easily from the fact that every element in a lacunary sequence with ratio at least 3 is greater than twice of sum of all previous elements in the sequence.

Simple fact

Let $(n_k)_{k=1}^{\infty}$ be sequence of positive integers such that $\frac{n_{k+1}}{n_k} \geq 3$ for every $k \in \mathbb{N}$ and for any set $A \subset \{n_k : k \in \mathbb{N}\}$, let us write $\widetilde{A}$ for the set defined as follows

$$\widetilde{A} = \left\{ \sum_{l=1}^{n} \varepsilon_l a_l : \varepsilon_l \in \{-1, 0, 1\}, a_l \in A \text{ (all distinct)}, n \in \mathbb{N} \right\}.$$

With these notions, if $A \cap B$ is finite, then $\widetilde{A} \cap \widetilde{B}$ is finite.

The proof of the non-separability of $\triangle(M(\mathbb{T}))$ II

We will also make use of a well-known observation due to Sierpiński.

Sierpiński

There exists uncountably many infinite subsets of positive integers such that the intersection of each two is finite.

Now, we recall a few facts on Riesz products: they are continuous probability measures on the circle of group of the following form

$$R(a_k, n_k) = \prod_{k=1}^{\infty}(1 + a_k \cos(n_k t)),$$

where this infinite product is meant as a weak* limit of finite products. From the construction of Riesz products we have (for simplicity we write $\mu = R(a_k, n_k)$ and $A = \{n_k : k \in \mathbb{N}\}$)

$$S(\mu) := \{n \in \mathbb{Z} : \widehat{\mu}(n) \neq 0\} = \widetilde{A}.$$

The proof of the non-separability of $\triangle(M(\mathbb{T}))$ III

For a sequence of natural numbers $(n_k)_{k=1}^{\infty}$ we assume $\frac{n_{k+1}}{n_k} \geq 3$ for every $k \in \mathbb{N}$ and from $(a_k)_{k=1}^{\infty}$ we demand $-1 < a_k \leq 1$ for $k \in \mathbb{N}$. We will use the following strong result on Riesz products.

Brown, Moran

If $(a_k)_{k=1}^{\infty}$ is a sequence of real numbers satisfying $-1 < a_k \leq 1$ for $k \in \mathbb{N}$ with the property

$$\sum_{k=1}^{\infty} |a_k|^n = \infty \text{ for all } n \in \mathbb{N}$$

then the Riesz product $R(a_k, n_k)$ has all convolution powers mutually singular.

It is an elementary result from the general theory of Banach algebras, that for Riesz products satisfying the assumptions of Theorem of Brown and Moran we have

$$\{z \in \mathbb{C} : |z| = 1\} \subset \sigma(R(a_k, n_k)).$$

The proof of the non-separability of $\triangle(M(\mathbb{T}))$ IV

In fact, a much stronger result is true.

Brown, Bailey, Moran

If $\mu \in M(\mathbb{T})$ is a hermitian probability measure with all convolution powers mutually singular then

$$\sigma(\mu) = \{z \in \mathbb{C} : |z| \leq 1\}.$$

Let us also take into account an easy fact concerning the Gelfand transforms of absolutely continous measures which follows from the partition of the Gelfand space into a disjoint sum of a Gelfand space of an ideal and of the set of all multiplicative-linear functionals vanishing on the ideal.

Gelfand transforms of absolutely continuous measures

If $\mu \in L^1(\mathbb{T})$ and $\varphi \in \triangle(M(\mathbb{T})) \setminus \mathbb{Z}$ then $\varphi(\mu) = 0$.

The proof of the non-separability of $\triangle(M(\mathbb{T}))$ V

We are ready now to prove the announced theorem.

Non-separability

$\triangle(M(\mathbb{T}))$ contains uncountably many pairwise disjoint open subsets. In particular, $\triangle(M(\mathbb{T}))$ is not separable.

We will prove the stronger assertion for the ideal $M_0(\mathbb{T})$ of measures with Fourier-Stieltjes coefficients vanishing at infinity. By the observation of Sierpiński there exists uncountably many subsets $\{A_\alpha\}_{\alpha\in\mathbb{R}}$ of $A = \{3^k : k \in \mathbb{N}\}$ with the property $A_{\alpha_1} \cap A_{\alpha_2}$ is finite for any $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 \neq \alpha_2$. For $\alpha \in \mathbb{R}$ we define the sequence $(a_k^\alpha)_{k=1}^{\infty}$ by the formula: $a_k^\alpha = \frac{1}{\ln((\text{the place of } k \text{ in } A_\alpha)+5)}$ for $k \in A_\alpha$ and $a_k^\alpha = 0$ otherwise. We assign to every A_α the Riesz product

$$\mu_\alpha := R(a_k^\alpha, n_k) = \prod_{k=1}^{\infty}(1 + a_k^\alpha \cos(3^k t)) \in M_0(\mathbb{T}).$$

The Proof of the non-separability of $\triangle(M(\mathbb{T}))$ VI

Then $S(\mu_\alpha) = \widetilde{A_\alpha}$. By the assumption and the Simple Fact we obtain $S(\mu_{\alpha_1}) \cap S(\mu_{\alpha_2})$ is finite for $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 \neq \alpha_2$. Now,

$$S(\mu_{\alpha_1} * \mu_{\alpha_2}) = S(\mu_{\alpha_1}) \cap S(\mu_{\alpha_2}) \text{ is finite.}$$

Therefore $\mu_{\alpha_1} * \mu_{\alpha_2}$ is a trigonometric polynomial.
Let us define for $\mu \in M(\mathbb{T})$:

$$\widehat{S}(\mu) = \{\varphi \in \triangle(M(\mathbb{T})) : \widehat{\mu}(\varphi) \neq 0\}.$$

Fix $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 \neq \alpha_2$ and take any $\varphi \in \triangle(M(\mathbb{T})) \setminus \mathbb{Z}$. Then $\varphi(\mu_{\alpha_1}) \cdot \varphi(\mu_{\alpha_2}) = \varphi(\mu_{\alpha_1} * \mu_{\alpha_2}) = 0$ because $\mu_{\alpha_1} * \mu_{\alpha_2} \in L^1(\mathbb{T})$.

The Proof of the non-separability of $\triangle(M(\mathbb{T}))$ VII

We can reformulate the last statement as

$$\widehat{S}(\mu_{\alpha_1}) \cap \widehat{S}(\mu_{\alpha_2}) \cap (\triangle(M(\mathbb{T})) \setminus \mathbb{Z}) = \emptyset \text{ for } \alpha_1 \neq \alpha_2.$$

Now, using Theorem of Bailey, Brown and Moran we know that there exists $z \in \mathbb{C} \setminus \mathbb{R}$ in $\sigma(\mu_\alpha)$ and, recalling that $\widehat{\mu_\alpha}(\mathbb{Z}) \subset \mathbb{R}$, we are able to find an open neighborhood U of z which does not intersect the real line. Since $\widehat{\mu_\alpha} : \triangle(M(\mathbb{T})) \mapsto \mathbb{C}$ is a continuous function we get $\widehat{\mu_\alpha}^{-1}(U) = T_\alpha$ is an open set contained in $\triangle(M(\mathbb{T})) \setminus \mathbb{Z}$ (as the Fourier-Stieltjes transform of μ_α is real-valued). On the other hand, $T_\alpha \subset \widehat{S}(\mu_\alpha)$ and hence $T_{\alpha_1} \cap T_{\alpha_2} = \emptyset$ for $\alpha_1 \neq \alpha_2$ which finishes the proof.

Extension of the theorem to other groups

With minor modifications, the same argument works for any compact group G.
Let us prove the theorem for $G = \mathbb{R}$. We have already built the family of hermitian probability measures with independent powers $\{\mu_\alpha\}_{\alpha\in\mathbb{R}} \subset M_0(\mathbb{T})$ such that $\mu_\alpha * \mu_\beta \in L^1(\mathbb{T})$ for $\alpha \neq \beta$. By well-known results from harmonic analysis we are able to define the family of measures $\{\nu_\alpha\}_{\alpha\in\mathbb{R}} \subset M_0(\mathbb{R})$ such that $\widehat{\nu_\alpha}$ agrees with $\widehat{\mu_\alpha}$ on $\mathbb{Z}$ and is extended linearly in the gaps. We have to verify that $\nu_\alpha * \nu_\beta \in L^1(\mathbb{R})$. Let $L : M(\mathbb{R}) \to M(\mathbb{T})$ be a mapping induced by the quotient homomorphism $\mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z} = \mathbb{T}$. On the level of Fourier-Stieltjes transforms we have $\widehat{L(\rho)} = \widehat{\rho}|_{\mathbb{Z}}$ which gives $L(\nu_\alpha * \nu_\beta) = \mu_\alpha * \mu_\beta \in L^1(\mathbb{T})$. The assertion follows from the fact that L sends singular measures to singular measures.
The extension to other locally compact Abelian groups is done via manipulations with the structure theorem for this class of groups.

Corollary

In the same manner we prove the following corollary which explains why determining the spectra of measures is so difficult task.

Corollary

There exists no countable set of multiplicative linear functionals on $M(\mathbb{T})$ such that the spectrum of any measure from $M(\mathbb{T})$ is a closure of the values of its Gelfand transform restricted to this set.

Open problem

We state the following problem of a much more general nature.

Open problem

Let A be a commutative unital Banach algebra such that there exists a countable family of multiplicative-linear functionals $\{\varphi_n\}_{n\in\mathbb{N}}$ satisfying

$$\sigma(x) = \overline{\{\varphi_n(x) : n \in \mathbb{N}\}} \text{ for all } x \in A.$$

Does it imply the separability of $\triangle(A)$?

Thank you for your attention!