On representations of "all but m" algebras

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Operator Gram matrix

Let $P_1, \dots, P_n$ be a family of projections in H, let $H_j = \operatorname{Im} P_j$, $j = 1, \dots, n$. Let $S_j \colon H_j \to H$ be isometric embeddings, so that $S_j S_j^* = P_j$, $S_j^* S_j = I_{H_j}$. Consider space $\tilde{H} = H_1 \oplus \cdots \oplus H_n$ and operator $J = (S_1, \dots, S_n) \colon \tilde{H} \to H$.

Definition (Yu.Samoilenko, A.Strelets, 2009; I.Feschenko, A.Strelets, 2012)

Operator $G = J^* J \colon \tilde{H} \to \tilde{H}$ is called operator Gram matrix, corresponding to the system of subspaces $(H; H_1, \dots, H_n)$.

Block entries of the operator Gram matrix are $(S_j^* S_k)_{j,k=1}^n$, therefore in the case where all P_j are one-dimensional projections we have $H_j = \mathbb{C}\langle e_j \rangle$, $\|e_j\| = 1$ and G is the Gram matrix of the system of vectors $(e_1, \dots, e_n)$.

Theorem (I.Feschenko, A.Strelets, 2012)

Operator Gram matrix possesses the following properties.

1. $G = G^*$, $G \geq 0$
2. *Diagonal entries of* G *are identity operators,* $G_{jj} = I_{H_j}$, $j = 1, \ldots, n$.
3. $G_{jk} = 0 \iff H_j \perp H_k$.

Let $Q_1, \ldots, Q_n$ be the projections on H_j in $\tilde{H}$.

Theorem (I.Feschenko, A.Strelets, 2012)

- *Family* $(P_1, \ldots, P_n)$ *in* H *is irreducible iff the family* $(G, Q_1, \ldots, Q_n)$ *is irreducible in* $\tilde{H}$.
- *Families* $(P_1, \ldots, P_n)$ *and* $(P'_1, \ldots, P'_n)$ *are unitary equivalent iff the corresponding families* $(G, Q_1, \ldots, Q_n)$ *and* $(G', Q'_1, \ldots, Q'_n)$ *are unitary equivalent.*

Inverse construction

Above, given a family of projections $P_1, \dots, P_n$ in H, we constructed the corresponding Gram operator $G \geq 0$ together with a family of projections $Q_1, \dots, Q_n$, such that

$$\sum_{j=1}^{n} Q_j = I, \quad Q_j G Q_j = Q_j$$

and showed that they carry information about the initial family.

Assume we have a family of projections $Q_1, \dots, Q_n$, in a Hilbert space $\tilde{H}$, $\sum_{k=1}^{n} Q_k = I$, and a bounded $B \geq 0$ in $\tilde{H}$ such that $Q_j B Q_j = Q_j$, $j = 1, \dots, n$. Then it is possible to recounstruct a family $P_1, \dots, P_n$, for which B would be the Gram operator matrix uniquely up to unitary equivalence.

All but one algebras

Consider $*$-algebra $\mathcal{P}_{abo,n}$ with generators $p, q_1, \ldots, q_n$ and relations

$$p^2 = p^* = p, \quad q_j^2 = q_j^* = q_j, \quad j = 1, \ldots, n,$$
$$q_1 + \cdots + q_n = e.$$

[N.Vasilevski, 1998]
A representation of this algebra is a family of projections $P, Q_1, \ldots, Q_n$ with $Q_1 + \cdots + Q_n = I$. For $n \geq 2$ the problem of unitary description of all representations is very complicated ($*$-wild).

All but one collections. One-dimensional projections

However, under the additional condition that

$$\dim \operatorname{Im} P = 1,$$

the description of all irreducible representations is a $*$-tame problem. The set of generic irreducible representations is parametrized by n-tuples of positive numbers $c_1, \dots, c_n$, for which

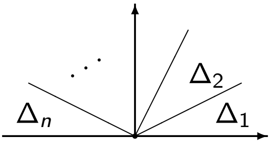
$$c_1 + \dots + c_n = 1,$$

and all irreducible representations are of dimesion $\leq n$.

Example: Bergmann space

Let Π be the upper complex half-plane. Functions from $L_2(\Pi)$ which are analytic inside Π form a subspace $\mathcal{A}^2(\Pi)$ (Bergmann subspace). Take P to be the projection onto $\mathcal{A}^2(\Pi)$ in $L_2(\Pi)$ (Bergmann projection).

Split Π into sectors $\Delta_1, \ldots, \Delta_n$ with origins at 0, $\bigcup_1^n \Delta_k = \Pi$,



and set $Q_j = M_{\chi_j}$, $j = 1, \ldots, m$. Then $Q_1 + \cdots + Q_n = I$, and the family $P, Q_1, \ldots, Q_n$ decomposes into irreducible representations with $\dim P \leq 1$.

Such families arise in the study of Töplitz operators (Vasilevski, 1998)

"All but m" families example: Bergmann type spaces

For each $k = 1, 2, \ldots,$ consider Bergmann type subspaces $\mathcal{A}_k^2, \mathcal{A}_{-k}^2 \subset L_2(\Pi)$ formed by functions f, for which $\frac{\partial^k}{\partial \bar{z}^k} f = 0$ or $\frac{\partial^k}{\partial z^k} f = 0$ correspondingly. Define

$$A_1^2 = \mathcal{A}_1^2 = \mathcal{A}^2, \quad A_k^2 = \mathcal{A}_k^2 \ominus \mathcal{A}_{k-1}^2, \quad k > 1,$$
$$A_{-1}^2 = \mathcal{A}_{-1}^2, \quad A_{-k}^2 = \mathcal{A}_{-k}^2 \ominus \mathcal{A}_{-k+1}^2, \quad k < -1.$$

and let $\mathcal{P}_k$ be the projection onto A_k in $L_2(\Pi)$. Then $\sum_1^\infty (\mathcal{P}_k + \mathcal{P}_{-k}) = I$ (Vasilevski, 1999).
Let $P_1, \ldots, P_m$ be a finite subset of $(\mathcal{P}_k)$. Then the family $Q_1, \ldots, Q_n, P_1, \ldots, P_m$ satisfies

$$Q_1 + \cdots + Q_n = I, \quad P_j P_k = 0, \quad j \neq k,$$

moreover, it decomposes into irreducible collections with $\dim P_j \leq 1$, $j = 1, \ldots, m$ (Yu.Karlovich, L.Pesoa, 2007)

Let H be a separable Hilbert space. We study the structure of families of projections $Q_1, \ldots, Q_n, P_1, \ldots, P_m$, such that

$$\sum_{i=1}^{n} Q_i = I, \quad P_j \perp P_k, \quad 1 \leq j \neq k \leq m, \tag{1}$$

up to a unitary equivalence. One can treat them as representations of the corresponding "all but m" $*$-algebra.
Of course, the problem of unitary description of *all* its representations is a $*$-wild problem. Below, we shall require some natural conditions which make the problem tame.

Gram matrix for "all but m" family

Consider isometric embeddings $S_j\colon \operatorname{Im} Q_j \to H$, $j = 1, \dots, n$, and isometric embeddings $T_i\colon \operatorname{Im} P_i \to H$, $i = 1, \dots, m$, so that

$$S_j^* S_j = I_{\operatorname{Im} Q_j}, \quad S_j S_j^* = Q_j, \quad T_i^* T_i = I_{\operatorname{Im} T_i}, \quad T_i T_I^* = P_i.$$

The operator Gram matrix G for the family of projections $P_1, \dots, P_m, Q_1, \dots, Q_n$ has the form:

$$G = \begin{pmatrix} I & \dots & 0 & T_1^* S_1 & \dots & T_1^* S_n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & I & T_m^* S_1 & \dots & T_m^* S_n \\ S_1^* T_1 & \dots & S_1^* T_m & I & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ S_n^* T_1 & \dots & S_n^* T_m & 0 & \dots & I \end{pmatrix}.$$

Nonocommutative decompositions of the identity

Denote by $H_i = \operatorname{Im} P_i$, $i = 1, \dots, m$, $H_0 = \bigoplus_{j=1}^{m} H_j$ and consider projections P'_j in H_0 onto subspaces H_j, $j = 1, \dots, m$.
Consider operators

$$B_j = \begin{pmatrix} T_1^* S_j \\ \vdots \\ T_m^* S_j \end{pmatrix} : \operatorname{Im} P_j \to H_0,$$

$$A_j = B_j B_j^* \colon H_0 \to H_0, \quad j = 1, \dots, n. \tag{2}$$

Proposition

The family $(P'_1, \dots, P'_m, A_1, \dots, A_n)$ in H_0 has the following properties.

1. $P'_1 + \dots + P'_m = I_{H_0}$, $A_1 + \dots + A_n = I_{H_0}$;
2. *If two families of orthogonal projections $(P_1, \dots, P_m, Q_1, \dots, Q_n)$ and $(\tilde{P}_1, \dots, \tilde{P}_m, \tilde{Q}_1, \dots, \tilde{Q}_n)$ are unitary equivalent, then the corresponding families of operators $(P'_1, \dots, P'_m, A_1, \dots, A_n)$ and $(\tilde{P}'_1, \dots, \tilde{P}'_m, \tilde{A}_1, \dots, \tilde{A}_n)$ are unitary equivalent.*

Remark

The inverse to the item (2) is wrong: it is easy to give examples of unitary nonequivalent families $(P_1, \dots, P_m, Q_1, \dots, Q_n)$ and $(\tilde{P}_1, \dots, \tilde{P}_m, \tilde{Q}_1, \dots, \tilde{Q}_n)$, which generate the same family $(P'_1, \dots, P'_m, A_1, \dots, A_n)$.

We say that subspaces $E \subset H$ and $K \subset H$ are in general position if $E \cap K = 0$, $E \cap K^\perp = 0$, $E^\perp \cap K = 0$. Denote $P_0 = P_1 + \cdots + P_m$.

Theorem

Let $P_1, \ldots, P_m, Q_1, \ldots, Q_n$ be an "all but m" family in H, and let $A_1, \ldots, A_n$ be the corresponding nonnegative operators in H_0, constructed as above, and let every pair P_0, Q_j, $j = 1, \ldots, n$, be in general position. Then

$\dim \operatorname{Im} P_0 = \dim \operatorname{Im} Q_j, \quad \ker A_j = 0, \quad j = 1, \ldots, n.$

Conversely, given projections $P'_1, \ldots, P'_m$, and nonnegative operators $A_1, \ldots, A_n$, in H_0 for which

$P'_1 + \cdots + P'_m = I, \quad A_1 + \cdots + A_n = I, \quad \ker A_j = 0, \quad j = 1, \ldots, n,$

then there exist a unique up to unitary equivalence "all but m" family $P_1, \ldots, P_m, Q_1, \ldots, Q_n$, for which P_0, Q_j, $j = 1, \ldots, n$, are in general position, and which give rise to the projections $P'_1, \ldots, P'_2$, and operators $A_1, \ldots, A_n$ as described above.

Irreducible representations

Let Λ be the set of all multi-indexes

$$\alpha = (j_1, k_1, j_2, \dots, k_{l-1}, j_l, k_l, j_1),$$
$$j_s = 1, \dots, m; \;\; k_s = 1, \dots, n; \;\; s = 1, \dots, l; \;\; l = 0, 1, \dots.$$

Consider the following family of operators C_α, $\alpha \in \Lambda$:

$$C_\alpha = P_{j_1} Q_{k_1} P_{j_2} \dots Q_{k_{l-1}} P_{j_l} Q_{k_l} P_{j_1}, \qquad (3)$$

Proposition

Let $Q_1, \dots, Q_n, P_1, \dots, P_m$ be an irreducible "all but m" family, for which $P_j \neq 0$, $j = 1, \dots, m$. Then $\dim P_j = 1$, $j = 1, \dots, m$ iff the operators C_α, $\alpha \in \Lambda$, form a commutative family.

Let $\Lambda_i \subset \Lambda$ be the subset of indexes, which start and end on i, $i = 1, \dots, m$, so that $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_m$.

Theorem

Let $P_1, \dots, P_2$, Q_1, $\dots$, Q_n be an irreducible "all but m family", for which
i) $P_i \neq 0$, $i = 1, \dots, m$;
ii) for every $j = 1, \dots, n$, the pair of projections $P_0 = P_1 + \dots + P_m$ and Q_j are in general position;
iii) the operators C_α, $\alpha \in \Lambda$, form a commutative family.
Then $C_\alpha = c_\alpha P_j$, $c_\alpha \in \mathbb{C}$, $\alpha \in \Lambda_j$, and the finite family of numbers

$$c_\alpha, \quad |\alpha| \in \{3, 5, 7\},$$

determines the projections $P_1, \dots, P_m$, Q_1, $\dots$, Q_n uniquely up to unitary equivalence.

Remark

1. In a general (not irreducible) situation, the joint specral decomposition of the commuting family of normal operators C_α*,* $\alpha \in \Lambda$ *gives a decomposition of the family* $P_1, \dots, P_m, Q_1, \dots, Q_n$ *into irreducible ones.*
2. The finite family of parameters described in the theorem above is redundant. For different concrete representations, different subsets of this family are sufficient.
3. The theorem does not provide the description of the set of all possible parameters. This set is determined by the conditions $A_1 + \cdots + A_n = I$*,* $A_j > 0$ *in* $\mathbb{C}^m$*.*