On real linear combinations of orthoprojections

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General form

$H$ — separable complex infinite-dimensional Hilbert space
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Orthoprojection $P^2 = P^* = P$

$$P_1 = \begin{pmatrix} \frac{\alpha}{\sqrt{\alpha - \alpha^2}} & \sqrt{\alpha - \alpha^2} \\ \sqrt{\alpha - \alpha^2} & 1 - \alpha \end{pmatrix} \in M_2(\mathbb{C})$$
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$$P_2 = \begin{pmatrix} A & \sqrt{A - A^2} U \\ U^* \sqrt{A - A^2} & U^* (I - A) U \end{pmatrix}$$

$\alpha \in [0, 1], \ 0 \leq A \leq I, \ U$ — unitary.
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Linear combinations $X = \alpha_1 P_1 + \alpha_2 P_2 + \cdots + \alpha_k P_k$

$X — ?$
Properties of sums, differences

\[ P_3 = \begin{pmatrix} \frac{A}{\sqrt{A-A^2}} & \sqrt{A-A^2} \\ \sqrt{A-A^2} & I - A \end{pmatrix} \quad P_4 = \begin{pmatrix} \frac{A}{-\sqrt{A-A^2}} & -\sqrt{A-A^2} \\ -\sqrt{A-A^2} & I - A \end{pmatrix} \]
Properties of sums, differences

\[
P_3 = \begin{pmatrix}
A & \sqrt{A - A^2} \\
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\end{pmatrix}
\]

\[
P_3 + P_4 = \begin{pmatrix}
2A & 0 \\
0 & 2I - 2A
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Properties of sums, differences

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\[ P_5 = \begin{pmatrix} I - A & \sqrt{A - A^2} \\ \sqrt{A - A^2} & \sqrt{A - A^2} \end{pmatrix} \]
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\[ P_5 = \begin{pmatrix} I - A & \sqrt{A - A^2} \\ \sqrt{A - A^2} & A \end{pmatrix} \]

\[ P_3 - P_5 = \begin{pmatrix} 2A - I & 0 \\ 0 & l - 2A \end{pmatrix} = \begin{pmatrix} T & 0 \\ 0 & -T \end{pmatrix} \]

\[ A \geq 0, \quad \|A\| \leq 1, \quad T^* = T, \quad \|T\| \leq 1 \]

Every bounded linear operator on $H$ is a complex linear combination of 257 orthoprojections.


Every bounded self-adjoint operator on a Hilbert space is a linear combination of 8 orthoprojections.

Theorem (P. A. Fillmore, JFA, 1969)

Every bounded self-adjoint operator on a Hilbert space is a real linear combination of 9 orthoprojections.

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On real linear combinations of orthoprojections

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Corollary

Every bounded operator on a Hilbert space is a complex linear combination of 8 orthoprojections.

Orthoprojection $Q$ — proper if $\dim \operatorname{Im} Q = \dim \operatorname{Im} (I - Q) = \infty$.

$P = Q_1 + Q_2$ or $P = Q_1 - Q_2$
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Corollary

*Every bounded self-adjoint operator $A$ on a Hilbert space is a real linear combination of 8 proper orthoprojections.*

Let $T \neq c \cdot I + K$. Every bounded linear operator on a Hilbert space is a complex linear combination of 14 operators unitarily equivalent $T$. 

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By additional reasoning, we get

Proposition (RVI, 2016)

Every bounded self-adjoint operator $A$ on a Hilbert space is a real linear combination of 5 proper orthoprojections.
Corollaries


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Proposition (S. Goldstein and A. Paszkiewicz, 1992, RVI, 2016)

Every bounded self-adjoint operator \( A \) on a Hilbert space is an integral linear combination of 5 orthoprojections.
Scheme of the proof

\[ A = \text{diag} (A_1, A_2) = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 \]
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K. Nishio, LAA, 1984

\[ \text{diag}(X^*X, \lambda I - XX^*) = aP_1 - bP_2, \; \lambda = a - b \]
A = \text{diag} (A_1, A_2) = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4

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\text{diag} (X^*X, \lambda I - XX^*) = aP_1 - bP_2, \lambda = a - b

If we take $X$ such that

\[ A_1 + A_2 - \lambda I = [X^*, X] = X^*X - XX^*. \]
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\[ \text{diag} \left( X^*X, \lambda I - XX^* \right) = aP_1 - bP_2, \quad \lambda = a - b \]

If we take \( X \) such that

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\[ \text{diag} \left( A_1, A_2 \right) - \text{diag} \left( X^*X, \lambda I - XX^* \right) = \text{diag} \left( T, -T \right) \]
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— when can we choose such an operator \( X \)?
For self-adjoint $A$, we shall denote by $\sigma_{\text{ess}}(A)$ the Weyl’s essential spectrum of $A$, that is the set

$$\bigcap_{K \in \mathcal{K}} \sigma(A + K)$$

where $\mathcal{K}$ is the set of all Hermitian compact operators. It consists of all limit points of $\sigma(A)$ and all eigenvalues of $A$ of infinite multiplicity.
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*Every Hermitian operator with 0 in the convex hull of its essential spectrum is a self-commutator*

$\lambda \in \sigma_{\text{ess}}(A_1 + A_2)$ — completes the proof.
Scheme of the proof. Finite matrix ???

\[ A = \text{diag} (A_1, A_2) = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 \]

Take

\[ \lambda \in \sigma_{\text{ess}} (A_1 + A_2) \]

Find \( X \) such that

\[ A_1 + A_2 - \lambda I = [X^*, X] = X^*X - XX^*. \]

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\[ \text{diag} (X^*X, \lambda I - XX^*) = aP_1 - bP_2, \ \lambda = a - b \]

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can be applied to finite matrices
Infinite dimensional space, two orthoprojections

\[ A = aP_1 + bP_2 \]
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**Proposition (N.L. Vasilevskii, I.M. Spitkovsky, 1981, K. Nishio, 84)**

Let \( a, b \in \mathbb{R} \setminus \{0\} \), \( P_1, P_2 \) be orthoprojections on \( H \). Then for every \( x \notin \{0, a, b, a + b\} \) the following implications hold

\[ x \in \sigma(aP_1 + bP_2) \iff (a + b - x) \in \sigma(aP_1 + bP_2). \]
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$H^4$, diag $(0.9 \cdot I, I, 1.00001 \cdot I, 1.01 \cdot I)$ is not a LC of two projections.
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$H^4$, \ diag $\begin{pmatrix} 0.9 \cdot I & I \\ 1.00001 \cdot I & 1.01 \cdot I \end{pmatrix}$ is not a LC of two projections

Except may be 2 points, a spectrum of linear combination of two orthoprojections is symmetric ($(a + b)/2$ is a symmetry point)
Proposition (RVI, 2016)

Let $K$ be a non-negative compact operator of infinite rank. Then $I - K$ is not a linear combination of three orthoprojections.

Corollary

Let $K$ be a non-negative compact operator of infinite rank. Then $I + K$ is not a linear combination of three orthoprojections.

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The operator of the form $I - K - iK$ is not a complex linear combination of four orthoprojections.
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$I - K = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3$

$\alpha_2 P_2 + \alpha_3 P_3 = I - K - \alpha_1 P_1$
Properties in use, interlace property

\[ I - K = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 \]

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\( P_i \) has to be proper
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\end{align*}

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**Proposition (H. Hochstadt, Proc. AMS, 1973)**

Let $K = \text{diag}(\mu_1, \mu_2, \mu_3, \ldots)$, where $\mu_1 > \mu_2 > \mu_3 > \ldots$, $\mu_n \to 0$, $n \to \infty$. For every rank one orthogonal projection $P$ and every $t > 0$, the set of eigenvalues $\gamma_1 \geq \gamma_2 \geq \gamma_3 \geq \ldots$ of $K + tP$ satisfy the interlace relation $\gamma_1 \geq \mu_1 \geq \gamma_2 \geq \mu_2 \geq \gamma_3 \geq \ldots$. 
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\[ 1 \text{ and } 1 - \alpha_1 \in \sigma_{\text{ess}}(\alpha_2 P_2 + \alpha_3 P_3) \]
\[ I - K = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 \]

\[ \alpha_2 P_2 + \alpha_3 P_3 = I - K - \alpha_1 P_1 \]

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\[ 1 \text{ and } 1 - \alpha_1 \in \sigma_{\text{ess}}(\alpha_2 P_2 + \alpha_3 P_3) \]
Proposition

Let \( a, b \in \mathbb{R} \setminus \{0\} \), \( P_1, P_2 \) be orthoprojections on \( H \). Then for every \( x \notin \{0, a, b, a + b\} \) the following implications hold

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In possible decompositions \( \alpha_1 + \alpha_2 + \alpha_3 = 2 \),
0 \( \alpha_i < 1 \) and \( K \) is of small norm

Proposition

Let \( P_1, P_2 \) be orthoprojections on \( H \) and \( 0 < \alpha_1 \leq \alpha_2 \). Then

\[
(\alpha_1 + \alpha_2 - c)P_{\text{Im}P_1 + \text{Im}P_2} \leq \alpha_1 P_1 + \alpha_2 P_2 \leq cI
\]
Theorem (Y. Nakamura, LAA, 1984)

Every Hermitian matrix from $M_n(\mathbb{C})$ is a real linear combination of 4 orthoprojections. For $n \leq 7$, it is a real linear combination of 3 orthoprojections.
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The scheme of the proof for infinite dimensional case can be directly applied to finite matrices in unitary space.

Proposition (RVI, 2016)

Let $A = \text{diag} (\mu_1, \ldots, \mu_4, \gamma_1 I_{18}, \ldots, \gamma_4 I_{18})$, where

$\mu_i = (1 - 10^{-10i}\theta)$, $\gamma_i = (1 + 10^{100(i-5)}\theta)$, $0 < \theta \leq 1$. Then $A$ is not a real linear combination of three orthoprojections.

$A$ is a small norm perturbation of $I$. 
Let $m \in \mathbb{N}$ and $m \geq 76$. The matrix $\text{diag}(A, \gamma_4 I_{m-76})$ is not a real linear combination of 3 orthoprojections.

It is interesting to know what is the maximal number $n$ for which every Hermitian $k \times k$ matrix is a real linear combination of three orthoprojections providing $k \leq n$. We suppose it is not greater than 25 or even 21.
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eigenvalues arranged $\lambda_i(A) \leq \lambda_{i+1}(A), \lambda_i(B) \leq \lambda_{i+1}(B)$

**Theorem (Weyl)**

*Let $A$ and $B$ be Hermitian $n \times m$ matrices and $\text{rank } B \leq k$. Then $\lambda_j(A + B) \leq \lambda_{j+k}(A) \leq \lambda_{j+2k}(A + B), j = 1, \ldots, n - 2k$.***
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**Theorem (Weyl)**

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$\lambda_j(A + B) \leq \lambda_{j+k}(A) \leq \lambda_{j+2k}(A + B)$, $j = 1, \ldots, n - 2k$.*

use frequently the monotonicity property for eigenvalues of Hermitian $n \times n$ matrices
$A \leq B \implies \lambda_i(A) \leq \lambda_i(B)$ for every $i = 1, \ldots, n$.
Operator inequalities
$(\alpha_1 + \alpha_2 - c)P_1 \text{Im}(P_1 + \text{Im} P_2) \leq \alpha_1 P_1 + \alpha_2 P_2 \leq cI$
C. Davis, J. Diximer, P. Halmosh, P. Fillmore, C. Pearcy, D. Topping, K.R. Davidson
A. Böttcher, I. Gohberg, I. M. Spitkovsky, N. Krupnik, B. Silbermann, T. Ehrhardt, N. Vasilevski, H. Bart
K. Matsumoto, Y. Nakamura, K. Nishio
S. Goldstein, A. Paszkiewicz
Yu. S. Samoǐlenko, S. Kruglyak, V. Ostrovskyi, I. S. Feshchenko
V. Kaftal, P. Casazza, G. Kutyniok, M. Fickus
M.-D. Choi, P. Y. Wu, C. K. Fong, G. J. Murthy, L.W. Marcoux
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Thank you!