On real linear combinations of orthoprojections

V. I. Rabanovich

Institute of Mathematics, Kyiv, Ukraine

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$$P_2 = \begin{pmatrix} A & \sqrt{A - A^2}\,U \\ U^*\sqrt{A - A^2} & U^*(I - A)U \end{pmatrix}$$

$\alpha \in [0, 1]$, $0 \leq A \leq I$, U — unitary.

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X — ?

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$$P_3 - P_5 = \begin{pmatrix} 2A - I & 0 \\ 0 & I - 2A \end{pmatrix} = \begin{pmatrix} T & 0 \\ 0 & -T \end{pmatrix}$$

$A \geq 0, \quad \|A\| \leq 1, \; T^* = T, \quad \|T\| \leq 1$

The beginning

Theorem (P. A. Fillmore, Acta Sci. Math., 1967)

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Theorem (C. Pearcy, D. Topping, Mich Math. J., 1967)

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Orthoprojection Q — *proper* if $\dim \mathrm{Im}\, Q = \dim \mathrm{Im}\,(I - Q) = \infty$.
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Every bounded self-adjoint operator A on a Hilbert space is a real linear combination of 8 proper orthoprojections.

Corollaries

Theorem (K.R. Davidson, L.W. Marcoux, J. Oper. Th, 2004)

Let $T \neq c \cdot I + K$. *Every bounded linear operator on a Hilbert space is a complex linear combination of* 14 *operators unitarily equivalent* T.

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Proposition (RVI, 2016)

Every bounded self-adjoint operator A on a Hilbert space is a real linear combination of 5 *proper orthoprojections.*

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Every bounded self-adjoint operator A on a Hilbert space is a real linear combination of 5 proper orthoprojections.

Proposition (S. Goldstein and A. Paszkiewicz, 1992, RVI, 2016)

Every bounded self-adjoint operator A on a Hilbert space is an integral linear combination of 5 orthoprojections.

Scheme of the proof

$$A = \operatorname{diag}(A_1, A_2) = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4$$

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$$\operatorname{diag}(A_1, A_2) - \operatorname{diag}(X^*X, \lambda I - XX^*) = \operatorname{diag}(T, -T)$$

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— when can we choose such an operator X?

Radjavi result

For self-adjoint A, we shall denote by $\sigma_{ess}(A)$ the Weyl's essential spectrum of A, that is the set

$$\bigcap_{K \in \mathcal{K}} \sigma(A + K)$$

where $\mathcal{K}$ is the set of all Hermitian compact operators. It consists of all limit points of $\sigma(A)$ and all eigenvalues of A of infinite multiplicity.

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Theorem (H. Radjavi, J. Math. Mech., 1966)

Every Hermitian operator with 0 in the convex hull of its essential spectrum is a self-commutator

$\lambda \in \sigma_{ess}(A_1 + A_2)$ — completes the proof.

Scheme of the proof. Finite matrix ???

$A = \mathrm{diag}\,(A_1, A_2) = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4$

Take

$$\lambda \in \sigma_{ess}(A_1 + A_2)$$

Find X such that

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can be applied to finite matrices

Infinite dimensional space, two orthoprojections

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Proposition (N.L. Vasilevski, I.M. Spitkovsky, 1981, K. Nishio, 84)

Let $a, b \in \mathbb{R} \setminus \{0\}$, P_1, P_2 *be orthoprojections on* H. *Then for every* $x \notin \{0, a, b, a+b\}$ *the following implications hold*

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Except may be 2 points, a spectrum of linear combination of two orthoprojections is symmetric ($(a+b)/2$ is a symmetry point)

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Corollary

The operator of the form $I - K - \boldsymbol{i}K$ is not a complex linear combination of 4 orthoprojections.

Properties in use, interlace property

$$I - K = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3$$

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Let $K = \mathrm{diag}\,(\mu_1, \mu_2, \mu_3, \dots)$, *where* $\mu_1 > \mu_2 > \mu_3 > \dots,$ $\mu_n \to 0$, $n \to \infty$. *For every rank one orthogonal projection* P *and every* $t > 0$, *the set of eigenvalues* $\gamma_1 \geq \gamma_2 \geq \gamma_3 \geq \dots$ *of* $K + tP$ *satisfy the interlace relation* $\gamma_1 \geq \mu_1 \geq \gamma_2 \geq \mu_2 \geq \gamma_3 \geq \dots$

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$$1 \text{ and } 1 - \alpha_1 \in \sigma_{ess}(\alpha_2 P_2 + \alpha_3 P_3)$$

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Symmetric property

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In possible decompositions $\alpha_1 + \alpha_2 + \alpha_3 = 2$, $0 < \alpha_i < 1$ and K is of small norm

Proposition

Let P_1, P_2 be orthoprojections on H and $0 < \alpha_1 \leq \alpha_2$. Then

$$(\alpha_1 + \alpha_2 - c)P_{\overline{\mathrm{Im}\,P_1 + \mathrm{Im}\,P_2}} \leq \alpha_1 P_1 + \alpha_2 P_2 \leq cI$$

Matrices

Theorem (Y. Nakamura, LAA, 1984)

Every Hermitian matrix from $M_n(\mathbb{C})$ is a real linear combination of 4 orthoprojections. For $n \leq 7$, it is a real linear combination of 3 orthoprojections.

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Proposition (RVI, 2016)

Let $A = \mathrm{diag}\,(\mu_1, \ldots, \mu_4, \gamma_1 I_{18}, \ldots, \gamma_4 I_{18})$, where $\mu_i = (1 - 10^{-10i}\theta)$, $\gamma_i = (1 + 10^{100(i-5)}\theta)$, $0 < \theta \le 1$. Then A is not a real linear combination of three orthoprojections.

A is a small norm perturbation of I

Corollary

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Let $m \in \mathbb{N}$ *and* $m \geq 76$. *The matrix* $\operatorname{diag}(A, \gamma_4 I_{m-76})$ *is not a real linear combination of* 3 *orthoprojections.*

It is interesting to know what is the maximal number n for which every Hermitian $k \times k$ matrix is a real linear combination of three orthoprojections providing $k \leq n$. We suppose it is not greater than **25** or even **21**.

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eigenvalues arranged $\lambda_i(A) \leq \lambda_{i+1}(A)$, $\lambda_i(B) \leq \lambda_{i+1}(B)$

Theorem (Weyl)

Let A and B be Hermitian $n \times m$ matrices and $\operatorname{rank} B \leq k$. Then $\lambda_j(A+B) \leq \lambda_{j+k}(A) \leq \lambda_{j+2k}(A+B)$, $j = 1, \ldots, n-2k$.

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use frequently the monotonicity property for eigenvalues of Hermitian $n \times n$ matrices

$A \leq B \Longrightarrow \lambda_i(A) \leq \lambda_i(B)$ for every $i = 1, \ldots, n$.

Operator inequalities

$(\alpha_1 + \alpha_2 - c)P_{\operatorname{Im} P_1 + \operatorname{Im} P_2} \leq \alpha_1 P_1 + \alpha_2 P_2 \leq cI$

Working team

C. Davis, J. Diximer, P. Halmosh, P. Fillmore C. Pearcy, D. Topping, K.R. Davidson

A. Böttcher, I. Gohberg, I. M. Spitkovsky, N. Krupnik, B. Silbermann, T. Ehrhardt, N. Vasilevski, H. Bart

K. Matsumoto, Y. Nakamura, K. Nishio

S. Goldstein, A. Paszkiewicz

Yu. S. Samoĭlenko, S. Kruglyak, V. Ostrovskyi, I. S. Feshchenko

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Thank you!