GROUPS AND OPERATORS
IN GOTHENBURG, AUGUST 2016

THE DIXMIER PROPERTY IN
OPERATOR ALGEBRAS

Rob Archbold (University of Aberdeen)

DEFN Unital C^*-algebra A has Dixmier property if

$\forall a \in A$, $\underbrace{\overline{co}\,\{u a u^* : u \in \mathcal{U}(A)\}}_{D_A(a)} \cap Z(A)$ is non-empty.

centre

Why hope? Consider semigrp Av of linear contractive $\alpha : A \to A$ of form

$$\alpha(x) = \sum_{j=1}^{n} t_j u_j x u_j^* \qquad (x \in A)$$

$$\alpha|_{D_A(a)} : D_A(a) \longrightarrow D_A(a) \qquad \text{affine}$$

IF $\exists$ fixed point $z \in D_A(a)$ then

$\forall u \in \mathcal{U}(A)$, $u z u^* = z$, $uz = zu$, $z \in Z(A)$.

$\underline{M_2(\mathbb{C})}$ Spse $a = a^*$. Wlog $a = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

$$\tfrac{1}{2}\left(1a1^* + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^*\right) = \tfrac{1}{2}\begin{bmatrix} \lambda_1+\lambda_2 & 0 \\ 0 & \lambda_2+\lambda_1 \end{bmatrix}$$
$$= tr(a)\,1$$

General $a = b + ic$. $\exists\ \alpha(b) = tr(b)\,1$.

$\alpha(a) = tr(b)1 + i\alpha(c)$. $\exists\ \beta(\alpha(c)) = tr(\alpha(c))1$

$(\beta\circ\alpha)(a) = tr(b)1 + i\,tr(c)1 = tr(a)\,1$.

$\underline{K(H) + \mathbb{C}1}$ E rank-1 proj, $E(e) = e$.

Form o-n basis $e, e_2, e_3, \dots$

Rank-1 proj's $E, E_2, E_3, \dots$

$$\frac{1}{n}\sum_{i=1}^{n} U_i E U_i^*$$

$= \frac{1}{n}(E + E_2 + \dots + E_n)$

has norm $\frac{1}{n}$.

$$U_i = \begin{array}{cc} & \begin{matrix} & & & i & & & \end{matrix} \\ \begin{matrix} \\ \\ \\ i \\ \\ \\ \\ \end{matrix} & \begin{bmatrix} 0 & & & 1 & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ 1 & & & 0 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & \ddots \end{bmatrix} \end{array}$$
$$\in \mathcal{U}(K(H) + \mathbb{C}1)$$

Dixmier's Theorem (1949)

Every von Neumann alg has Dixmier property.

Applications

1) DP used in cohomology of vN alg. KLR, Jo, SS et al

2) Finite vN alg's have the singleton Dix prop (SDP):

$\forall a \in A, \quad D_A(a) \cap Z(A) = \{T(a)\}$

where $T : A \to Z(A)$ is unique centre-valued trace.

Spse C^*-alg A has DP and $J \lhd A$. $q : A \to A/J$.

$a \in A$: $\quad \exists z \in \overline{co}\{uau^* : u \in \mathcal{U}(A)\} \cap Z(A)$

$q(z) \in \overline{co}\{vq(a)v^* : v \in \mathcal{U}(A/J)\} \cap Z(A/J)$

So A/J has DP and $Z(A/J) \underset{\oplus}{=} \dfrac{Z(A)+J}{J}$.

[$J + \mathbb{C}1$ also has DP, and $\exists$ lifting thm subject to $\oplus$.]

EX $A = K(H) + \mathbb{C}e + \mathbb{C}f$, $e + f = 1$ inf dim proj's.

$Z(A) = \mathbb{C}1$. $Z\left(\frac{A}{K(H)}\right) \cong \mathbb{C} \oplus \mathbb{C}$. So A fails DP.

Haagerup – Zsido Thm (1984)

Let $1 \in A$, simple C^*-algebra. Then

A has DP $\Leftrightarrow$ A has at most one tracial state.

$C^*_\lambda(G)$ $\quad \lambda : G \to B(\ell^2(G))$, $\quad \lambda_g(f)(h) = f(g^{-1}h)$

left reg repn

$C^*_\lambda(G) := \overline{\text{span}}\{\lambda_g : g \in G\}$

Trace $\tau : C^*_\lambda(G) \to \mathbb{C}$, $\quad \tau(a) = \langle a\delta_e, \delta_e \rangle$

$\tau(a^*a) = 0 \Rightarrow \|a\delta_e\|^2 = 0 \Rightarrow a\delta_e = 0 \Rightarrow a = 0.$

Powers (1975) $\quad G = \mathbb{F}_2$. $\quad \forall a \in C^*_\lambda(\mathbb{F}_2)$,

$\overline{co}\{uau^* : u \in \mathcal{U}(C^*_\lambda(\mathbb{F}_2))\} \cap \mathbb{C}1$ is not empty.

Spse $\{0\} \neq J \lhd C^*_\lambda(\mathbb{F}_2)$. Let $0 \neq a \in J$.

$\exists \; t1 \in \overline{co}\{ua^*au^* : u \ldots\} \subseteq J$

$t = \tau(t1) \in \{\tau(a^*a)\}$ so $t \neq 0$.

$C^*_\lambda(\mathbb{F}_2)$ is simple, has SDP and unique tracial state.

Same for many other G.

de la Harpe, BLMS survey 2007.

$$\begin{array}{lll} \text{DP} & \Longrightarrow & (a)\ C^*_\lambda(G) \text{ simple} \\ (\text{with } Z = \mathbb{C}1) & \Longrightarrow & (b)\ C^*_\lambda(G) \text{ has unique tracial state} \\ & & \Downarrow \\ & & (c)\ \mathrm{Rad}(G) = \{e\} \end{array}$$

($\{e\}$ is the only normal amenable subgp)

$$
\begin{tikzcd}[column sep=large, row sep=large]
\text{DP (with } Z=\mathbb{C}1) \arrow[r, shift right, Rightarrow] & (a)\ C^*_\lambda(G)\ \text{simple} \arrow[l, shift right, Rightarrow, "\text{H 2015, K 2015}"'] \arrow[r, Leftrightarrow, "\text{KK}", "2014"'] \arrow[dd, bend right=60, Rightarrow] & \begin{array}{l}G \text{ has a top free} \\ \text{bndry action on} \\ \text{some compact } X\end{array} \\
& (b)\ C^*_\lambda(G)\ \text{has unique tracial state} \arrow[u, Rightarrow, "\times" marking, "\text{Le Boudec 2015}"'] \arrow[d, shift right, Rightarrow] & \\
& (c)\ \mathrm{Rad}(G)=\{e\} \arrow[u, shift right, Rightarrow, "\text{BKKO 2014}"'] &
\end{tikzcd}
$$

Note $(a) \xRightarrow{\text{BKKO}} (a)+(b) \xRightarrow{\text{H-Zsido}}$ DP (with $Z=\mathbb{C}1$),

but Haagerup and Kennedy both give direct proof with more detailed info (enough to use $\lambda(G)$ rather than full unitary group).

Ozawa (2013)

Theorem 1 Let $1 \in A$ and assume tracial state space $T(A)$ is non-empty. TFAE:

(i) $T(A/M)$ is non-empty $\forall$ max ideals M of A.

(ii) For every $\varepsilon > 0$ and $a \in A$ such that

$$\sup_{\tau \in T(A)} |\tau(a)| < \varepsilon,$$

$\exists$ $k \geq 1$ and $u_1, \ldots, u_k \in \mathcal{U}(A)$ such that

$$\left\| \frac{1}{k} \sum_{i=1}^{k} u_i a u_i^* \right\| < \varepsilon$$

e.g. A exact $+$ $sr(A) = 1 \xRightarrow[2006]{\text{Murphy}}$ (i)

Theorem 2 Let $1 \in A$. TFAE

(a) A is amenable (nuclear) and (i) holds.

(b) A is symmetrically amenable

(c) A has a symm. approx. diagonal (Δ_n) in

$$\left\{ \sum_i x_i^* \otimes x_i \in A \otimes_{\mathbb{C}} A : \sum_i \|x_i\|^2 \leq 1 \right\}$$

Theorem 3 (AST) Let A be a unital C^*-algebra.

$$A \text{ has SDP} \iff \begin{cases} \text{(i)} & T(A/M) \text{ non-empty } \forall \text{ max ideals } M \\ \text{(ii)} & r : T(A) \to S(Z(A)) \text{ is bijective.} \end{cases}$$

FACT (Dixmier)

$$\left.\begin{array}{r} A \text{ has DP} \\ \text{and } Z(A) = \mathbb{C}1 \end{array}\right\} \Longrightarrow A \text{ has a unique max ideal}$$

Theorem 4 (AST) Let A be a unital C^*-algebra.

(I) $$\left.\begin{array}{r} A \text{ has SDP} \\ \text{and } Z(A) = \mathbb{C}1 \end{array}\right\} \iff \left\{\begin{array}{l} A \text{ has a unique tracial state } \tau \\ \text{and} \\ \qquad \{a \in A : \tau(a^*a) = 0\} \\ \text{is the unique max ideal of } A. \end{array}\right.$$

(II) $$\left.\begin{array}{r} A \text{ has DP} \\ \text{and } Z(A) = \mathbb{C}1 \end{array}\right\} \iff \left\{\begin{array}{l} A \text{ has a unique max ideal } M, \\ A \text{ has at most one tracial state,} \\ M \text{ has no tracial states.} \end{array}\right.$$

(III) $$\left.\begin{array}{r} A \text{ has DP but } \underline{\text{not}} \text{ SDP} \\ \text{and } Z(A) = \mathbb{C}1 \end{array}\right\} \iff \left\{\begin{array}{l} A \text{ has a unique max ideal} \\ \text{and no tracial states} \end{array}\right.$$

DEFN A has the uniform Dixmier property (UDP) if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ such that $\forall \|a\| \leq 1$ $\exists u_i, \lambda_i$ $(1 \leq i \leq N)$ with $d\left(\sum_{i=1}^{N} \lambda_i u_i a u_i^*, Z(A)\right) \leq \varepsilon$.

e.g. vN algebras and $C^*_\lambda(\mathbb{F}_2)$ have UDP.

Consider unital, simple A with unique trace τ.
A has DP (in fact, SDP) by Haagerup-Zsido.
Consider an ultrapower A_ω. $\tau \rightsquigarrow \tau_\omega \in T(A_\omega)$.

$$J := \{[(a_n)] \in A_\omega : \lim_\omega \tau(a_n^* a_n) = 0\} \in \mathrm{Max}(A_\omega).$$

Theorem 5 (AST)

A has UDP $\iff$ A_ω has DP $\overset{\text{Th4}}{\iff}$ $\begin{cases} T(A_\omega) = \{\tau_\omega\} \text{ and} \\ J \text{ is unique max ideal of } A_\omega. \end{cases}$

Cor 6 Examples of Villadsen and of L. Robert give simple A with unique τ such that A fails UDP.

Cor 7 Suppose A is unital, simple and has unique tracial state. If A has strict comparison of positive elements then A has the UDP.

Approximately finite-dimensional (AF) C^*-algebras.

Suppose $A = \overline{\bigcup_{k\geq 1} A_k}$ where $1_A \in A_k \uparrow$, A_k fin-dim with (unique) centre-valued trace $R_k : A_k \to Z(A_k)$.

Theorem 8 (AST) TFAE

(i) A has the UDP

(ii) A has the DP

(iii) A has the SDP

(iv) $\forall k \geq 1$ and $a \in A_k$, $(R_m(a))_{m\geq k}$ is convergent.

(v) $\forall k \geq 1$ and $a \in Z(A_k)$, $(R_m(a))_{m\geq k}$ is cvgt.

When these conditions hold, A has a (unique) centre-valued trace $R : A \to Z(A)$ and $R(a)$ is the limit in (iv) and in (v).

Remarks (a) In some examples, condition (v) can be checked from a Bratteli diagram.

(b) The equivalent conditions can fail to hold. e.g. $\exists$ simple, unital, AF algebras with more than one tracial state (and hence DP fails).