Operator ultra-amenability

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Just as a reminder...

Definition (B. E. Johnson, 1972)

A Banach algebra $\mathfrak{A}$ is called amenable if, for every Banach $\mathfrak{A}$-bimodule E, every derivation $D : \mathfrak{A} \to E^*$, is inner.

Theorem (B. E. Johnson, 1972)

The following are equivalent for a locally compact group G:

1. *$L^1(G)$ is an amenable Banach algebra;*
2. *the group G is amenable.*

Ultrapowers of Banach spaces

Definition

Let E be a Banach space, let $\mathbb{I} \neq \varnothing$, and let $\mathcal{U}$ be an ultrafilter over $\mathbb{I}$. Set

$$\ell^\infty(\mathbb{I}, E) := \left\{ (x_i)_{i \in \mathbb{I}} : \sup_{i \in \mathbb{I}} \|x_i\| < \infty \right\}$$

and

$$\mathcal{N}_\mathcal{U} := \left\{ (x_i)_{i \in \mathbb{I}} : \lim_{i \in \mathcal{U}} \|x_i\| = 0 \right\}.$$

Then $(E)_\mathcal{U} := \ell^\infty(\mathbb{I}, E)/\mathcal{N}_\mathcal{U}$ is called an ultrapower of E.

...and of Banach algebras

Example

Let $\mathcal{U}$ be an arbitrary ultrafilter over an arbitrary index set. Then:

1. if $\mathfrak{A}$ is a Banach algebra, then so is $(\mathfrak{A})_{\mathcal{U}}$;
2. if $\mathfrak{A}$ is a C^*-algebra, then so is $(\mathfrak{A})_{\mathcal{U}}$.

Definition (M. Daws, 2009)

$\mathfrak{A}$ is called ultra-amenable if $(\mathfrak{A})_{\mathcal{U}}$ is amenable for every ultra-filter $\mathcal{U}$ (over an arbitrary index set).

Contractibility, amenability, and ultra-amenability

Proposition

$$\mathfrak{A} \text{ contractible} \implies \mathfrak{A} \text{ ultra-amenable} \implies \mathfrak{A} \text{ amenable.}$$

Theorem

For a C^*-algebra $\mathfrak{A}$:

1. *$\mathfrak{A}$ is contractible iff $\dim \mathfrak{A} < \infty$;*
2. *$\mathfrak{A}$ is ultra-amenable iff $\mathfrak{A}$ is subhomogeneous (M. Daws, 2009);*
3. *$\mathfrak{A}$ is amenable iff $\mathfrak{A}$ is nuclear.*

Ultra-amenability of $L^1(G)$

Theorem (M. Daws, 2009)

Let G be such that $L^1(G)$ is ultra-amenable, and suppose that any of the following holds:

1. *G is abelian;*
2. *G is compact;*
3. *G is discrete.*

Then G is finite.

Conjecture

For any G, $L^1(G)$ is ultra-amenable iff G is finite.

Completely contractive Banach algebras

Definition

An algebra $\mathfrak{A}$ that is also an operator space such that multiplication in $\mathfrak{A}$ is completely contractive is called a completely contractive Banach algebra.

Example

1. If $\mathfrak{A}$ is a Banach algebra, then $\max \mathfrak{A}$ is a completely contractive Banach algebra.
2. Every closed subalgebra of $\mathcal{B}(\mathfrak{H})$ is a completely contractive Banach algebra.
3. For every locally compact group G, its Fourier algebra $A(G) = \mathrm{VN}(G)_*$ is a completely contractive Banach algebra.

Operator amenability, I

Definition (Z.-J. Ruan, 1995)

A completely contractive Banach algebra $\mathfrak{A}$ is called operator amenable if, for every completely bounded Banach $\mathfrak{A}$-bimodule E, every completely bounded derivation $D : \mathfrak{A} \to E^*$, is inner.

Proposition

The following are equivalent for a Banach algebra $\mathfrak{A}$:

1. $\mathfrak{A}$ *is amenable;*
2. $\max \mathfrak{A}$ *is operator amenable.*

Operator amenability, II

Theorem (Z.-J. Ruan, 1995)

The following are equivalent for a C^*-algebra $\mathfrak{A}$:

1. *$\mathfrak{A}$ is operator amenable;*
2. *$\mathfrak{A}$ is amenable.*

Theorem (Z.-J. Ruan, 1995)

The following are equivalent for G:

1. *$A(G)$ is operator amenable;*
2. *G is amenable.*

Theorem (B. E. Forrest & V. Runde, 2005)

$A(G)$ is amenable iff G has an abelian subgroup of finite index.

Operator ultra-amenability—definition and generalities

Definition

A completely contractive Banach algebra $\mathfrak{A}$ is called operator ultra-amenable if $(\mathfrak{A})_{\mathcal{U}}$ is operator amenable for every ultrafilter $\mathcal{U}$.

Proposition

For every Banach space E, we have $\max(E)_{\mathcal{U}} \cong (\max E)_{\mathcal{U}}$ completely isometrically.

Corollary

The following are equivalent for a Banach algebra $\mathfrak{A}$:

1. *$\mathfrak{A}$ is ultra-amenable;*
2. *$\max \mathfrak{A}$ is operator ultra-amenable.*

The C^*-case

Proposition

The following are equivalent for a C^*-algebra $\mathfrak{A}$:

1. *$\mathfrak{A}$ is operator ultra-amenable;*
2. *$\mathfrak{A}$ is ultra-amenable;*
3. *$\mathfrak{A}$ is subhomogeneous.*

Operator ultra-amenability of $A(G)$, I

Question

For which G is $A(G)$ operator ultra-amenable?

Conjecture

$A(G)$ is operator ultra-amenable iff G is finite.

Lemma

Let G be such that $A(G)$ is operator ultra-amenable, and let H be a closed subgroup of G. Then $A(H)$ is operator ultra-amenable.

Operator ultra-amenability of $A(G)$, II

Proof.

The restriction map

$$A(G) \to A(H), \quad f \mapsto f|_H$$

is a complete quotient map, as is thus

$$(A(G))_{\mathcal{U}} \to A(H))_{\mathcal{U}}, \quad (f_i)_{\mathcal{U}} \mapsto (f_i|_H)_{\mathcal{U}},$$

for any ultrafilter $\mathcal{U}$. □

Corollary

Let G be such that $A(G)$ is operator ultra-amenable. Then G contains no infinite abelian subgroup.

Operator ultra-amenability of $A(G)$, III

Proposition

Let H be an infinite closed subgroup of G such that one of the following holds:

1. *H is compact;*
2. *H is connected.*

Then $A(G)$ is not operator ultra-amenable.

Proof.

It is enough to show that $A(H)$ is not operator ultra-amenable. If H is compact, it contains an infinite abelian subgroup (E. I. Zelmanov, 1992). This proves the claim.

Operator ultra-amenability of $A(G)$, IV

Proof (continued).

If H is connected, there are closed abelian subgroups $H_1, \ldots, H_n$ of G and a compact subgroup K of G such that

$$H = H_1 \cdots H_n K.$$

As $H_1, \ldots, H_n$ and K must be finite, H must be finite. □

Theorem

Let G be such that $A(G)$ is operator ultra-amenable. Then G is discrete, amenable, and contains no infinite abelian subgroup.

Operator ultra-amenability of $A(G)$, V

Proof.

Let G_e be the component of G containing the identity.
Then G_e must be finite and thus trivial.
Hence, G is totally disconnected.
It follows that G has a neighborhood base of the identity consisting of compact open subgroups (H. Yamabe, 1953).
Each of these subgroups must be finite.
Hence, G is discrete. □

Examples supporting our conjecture

Example

Suppose that $A(G)$ is operator ultra-amenable and that any of the following holds:

1. G is locally finite;
2. G is elementary amenable;
3. G is linear;
4. G has polynomial growth.

Then G is finite.