Property (T) and Haagerup property for quantum groups – a global point of view
based on joint work with M. Daws, P. Fima, A. Viselter and S. White

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IMPAN
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Unitary representations

\( G \) – locally compact group

A (unitary, strongly continuous) representation \( \pi \) of \( G \) on a Hilbert space \( H \)

- contains an invariant vector if
  \[
  \exists \xi \in H, \|\xi\| = 1 \quad \forall g \in G \quad \pi(g)\xi = \xi;
  \]

- contains almost invariant vectors if
  \[
  \exists \xi_i \in H, \|\xi_i\| = 1 \quad \forall g \in G \quad \pi(g)\xi_i - \xi_i \to 0
  \]
  (uniformly on compact subsets)

- is mixing if
  \[
  \forall \xi, \eta \in H \quad \langle \xi, \pi(\cdot)\eta \rangle \in C_0(G);
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- is ergodic if it does not contain an invariant vector;

- is weakly mixing if \( \pi \otimes \bar{\pi} \) is ergodic.
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**General notations**

\( \mathbb{G} \) – a locally compact quantum group à la Kustermans-Vaes

\( L^\infty(\mathbb{G}) \) – a von Neumann algebra equipped with the *coproduct*

\[ \Delta : L^\infty(\mathbb{G}) \rightarrow L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{G}) \]

carrying all the information about \( \mathbb{G} \)

\( C_0(\mathbb{G}) \) – the corresponding (reduced) \( C^* \)-object, \( C_b(\mathbb{G}) := M(C_0(\mathbb{G})) \)

\( C_0^u(\mathbb{G}) \) – the universal version of \( C_0(\mathbb{G}) \)

\( L^2(\mathbb{G}) \) – the GNS Hilbert space of the *right invariant Haar weight* on \( \mathbb{G} \)

\( W^\mathbb{G} \in B(L^2(\mathbb{G}) \otimes L^2(\mathbb{G})) \) – the multiplicative unitary associated to \( \mathbb{G} \):

\[ \Delta(f) = W^\mathbb{G}(f \otimes 1)(W^\mathbb{G})^*, \quad f \in L^\infty(\mathbb{G}). \]

Note the inclusions

\[ C_0(\mathbb{G}) \subset C_b(\mathbb{G}) \subset L^\infty(\mathbb{G}) = C_0(\mathbb{G})'' \]
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Dual quantum groups

Each LCQG $G$ admits the dual LCQG $\widehat{G}$.

$L^\infty(\widehat{G})$, $C_0(\widehat{G})$ – subalgebras of $B(L^2(G))$

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Further properties of LCQGs

**Definition**

A locally compact quantum groups $\mathbb{G}$ is

- **compact** if $C_0(\mathbb{G})$ is unital (equivalently the Haar weights are finite);
- **discrete** if $\hat{\mathbb{G}}$ is compact;
- **unimodular** if the left and right Haar weights coincide;
- **of Kac type** if the so-called scaling group is trivial (the antipode is a bounded map);
- **amenable** if $L^\infty(\mathbb{G})$ admits a bi-invariant mean;
- **coamenable** if the universal and reduced algebras $C_0(\mathbb{G})$ and $C_0^u(\mathbb{G})$ are naturally isomorphic;
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Some examples of locally compact quantum groups

- locally compact groups (all coamenable);
- duals of locally compact groups (all amenable);
- quantum deformations of classical Lie groups: for example $SU_q(2)$, quantum $ax + b$, $E_q(2)$ (amenable and coamenable, usually not Kac);
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Representations of LCQGs

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$$(\Delta \otimes \iota)(U) = U_{13} U_{23}.$$ 

The operators $(\iota \otimes \omega_{\xi,\eta})(U) \in C_b(\mathbb{G})$, where $\xi, \eta \in H$, are called coefficients of $U$.

Representations of $\mathbb{G}$ are in a 1-1 correspondence with $C^*$-representations of $C_0^u(\hat{\mathbb{G}})$.

One can also tensor representations of $\mathbb{G}$ ($U \boxtimes V$), take direct sums ($U \oplus V$) and pass to a contragredient representation $U^c$. 
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Representations of LCQGs – continued

Definition

A representation $U$ of $G$ is mixing if all its coefficients belong to $C_0(G)$. It has almost invariant vectors if there exists a net of unit vectors $(\xi_i)_{i \in I}$ such that for all $a \in C_0(G)$

$$U(a \otimes \xi_i) - a \otimes \xi_i \rightarrow 0$$

– equivalently for all $b \in C_0^u(\hat{G})$

$$\phi_U(b)\xi_i - \hat{\epsilon}(b)\xi_i \rightarrow 0$$ strictly.

The multiplicative unitary $W^G$ plays the role of the left regular representation of $G$ on $L^2(G)$; it is mixing.
Definitions and first facts

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Space of representations

$\mathcal{G}$ – second countable locally compact quantum group, $H$ – fixed infinite dimensional separable Hilbert space. Then $\text{Rep}_G(H)$ is a Polish space with a natural (‘point-weak’) topology. It is equipped with two natural operations: **direct sum** (finite or countable) and **tensoring** (after we fix some unitary identifications of $H \otimes H$ with $H$, etc.).

**Lemma (DFSW)**

Suppose $\mathcal{R} \subset \text{Rep}_G(H)$

1. is stable under unitary equivalence;
2. is stable under tensoring with any $V \in \text{Rep}_G(H)$;
3. contains a representation with almost invariant vectors.

Then $\mathcal{R}$ is dense in $\text{Rep}_G(H)$. 
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Theorem (DFSW)

A second countable locally compact quantum group $\mathbb{G}$ has HAP if and only if the set of mixing representations is dense in $\text{Rep}_\mathbb{G}(H)$.

Proof.

$\Leftarrow$

Approximate the trivial representation with mixing ones and take their direct sum (still mixing!).

$\Rightarrow$

Use the lemma with $\mathcal{R} –$ mixing representations.

Ideas go back to the work of Halmos for $\mathbb{Z}$. 
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Theorem (Kerr-Pichot, 2012)

Let $G$ – classical locally compact group, second countable. Then $G$ does not have Property (T) if and only if weakly mixing representations are dense in $\text{Rep}_G(H)$.

We want to show the same for (a class of) quantum groups.

Recall: $U \in \text{Rep}_G(H)$ is weakly mixing if $U \overline{\otimes} U^c$ is ergodic.

- when is the class of weakly mixing representations stable under tensoring?
- when not (T) means that there is a weakly mixing representation with almost invariant vectors?
Theorem (Kerr-Pichot, 2012)

Let $G$ – classical locally compact group, second countable. Then $G$ does not have Property (T) if and only if weakly mixing representations are dense in $\text{Rep}_G(\mathcal{H})$.

We want to show the same for (a class of) quantum groups. Recall: $U \in \text{Rep}_G(\mathcal{H})$ is weakly mixing if $U \tilde{\otimes} U^c$ is ergodic.

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(T) via ‘typical’ representations

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Weak mixing representations revisited

Classically: $U$ is not weakly mixing if and only if $U \uplus U^c$ contains an invariant vector if and only if $U$ contains a finite dimensional subrepresentation. More generally: $U \uplus V^c$ contains a fixed vector if and only if $U$ and $V$ contain the same finite dimensional subrepresentation.

Lemma (Chen+Ng 2015, see also Kyed+Soltan, Viselter)

If $G$ is of Kac type, then a representation of $G$ is weakly mixing if and only if it does not contain a finite dimensional subrepresentation; hence then the class of weakly mixing representations is stable under tensoring.
Weak mixing representations with almost invariant vectors

Let us contradict the statement: there is a representation of $\mathbb{G}$ which is weakly mixing and has almost invariant vectors.

**Definition**

$\mathbb{G}$ has Property (T)$^{1,1}$ (of Bekka and Valette) if for every representation $U$ of $\mathbb{G}$ with almost invariant vectors $U \Upsilon U^c$ has a fixed vector.

Obviously $(T) \implies (T)^{1,1}$.

**Theorem (Bekka and Valette)**

For classical groups $(T) \iff (T)^{1,1}$.

This gives the result of Kerr and Pichot.
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Discrete quantum groups with low duals

Γ – discrete quantum group. Then

\[ c_0(\Gamma) = \bigoplus_{i \in I} M_{n_i} \]

**Definition**

Γ as above has a low dual if \( \sup_{i \in I} n_i < \infty \).

In other words, we have a uniform bound on the size of irreducible representations of the compact quantum group dual to Γ.
Main theorem

**Theorem (DSV, 2016)**
Let $\Gamma$ – discrete unimodular second countable quantum group with a low dual. Then $\Gamma$ has Property (T) if and only if it has Property (T)$^{1,1}$

**Corollary (DSV, 2016)**
Let $\Gamma$ – discrete unimodular second countable quantum group with a low dual. Then $\Gamma$ does not have (T) if and only if weakly mixing representations form a dense $G_\delta$-set in $\text{Rep}_\Gamma(H)$. 
Main theorem – ingredients of the proof

**Problem**: assume \( \Gamma \) does not have \((T)\). Construct a representation \( U \) of \( \Gamma \) with almost invariant vectors such that \( U \dagger U^c \) contains no invariant vector.

**Idea** (Jolissaint): use non \((T)\) of \( \Gamma \) to construct a semigroup of states on the algebra \( C(\hat{\Gamma}) \) with particular properties.
Main theorem – ingredients of the proof

**Problem**: assume $\Gamma$ does not have (T). Construct a representation $U$ of $\Gamma$ with almost invariant vectors such that $U \oplus U^c$ contains no invariant vector.

**Idea** (Jolissaint): use non (T) of $\Gamma$ to construct a semigroup of states on the algebra $C(\hat{\Gamma})$ with particular properties.
Main theorem – ingredients of the proof

Actual ingredients:

- development of the notion of Kazhdan pairs for quantum groups with Property (T);
- application of this to showing that if a locally compact $\mathbb{G}$ is of Kac type and does not have (T) then one can find a net of positive-definite normalised positive elements in $C_b(\mathbb{G})$ which converge to 1 strictly, but not in norm;
- use of Yukio Arano’s work on central Property (T) to show that for discrete unimodular case one can choose the elements above in the centre;
- construction of a strongly unbounded symmetric generating functional $L$ on $\hat{\Gamma}$;
- using $L$ to generate a convolution semigroup of states $\mu_t$ on $C^u(\hat{\Gamma})$;
- building out of $\mu_t$ ‘symmetric’ GNS representations of $C^u(\hat{\Gamma})$ (thus self-contragredient representations $U_t$ of $\Gamma$);
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Other consequences of \((T) \iff (T)^{1,1}\)

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Let \(\Gamma\) – discrete unimodular second countable quantum group with a low dual. Then the following are equivalent:

1. \(\Gamma\) has Property (T);
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3. every ergodic action of \(\Gamma\) on a von Neumann algebra preserving a faithful normal state is strongly operator ergodic (i.e. asymptotically invariant nets of elements in the von Neumann algebra are trivial).

These generalize classical results of Li, Ng, Connes and Weiss.
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