Property (T) and Haagerup property for quantum groups – a global point of view

based on joint work with M. Daws, P. Fima, A. Viselter and S. White

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IMPAN

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Unitary representations

G – locally compact group

A (unitary, strongly continuous) representation π of G on a Hilbert space H

- **contains an invariant vector** if

$$\exists_{\xi \in \mathsf{H}, \|\xi\|=1} \forall_{g \in G} \; \pi(g)\xi = \xi;$$

- **contains almost invariant vectors** if

$$\exists_{\xi_i \in \mathsf{H}, \|\xi_i\|=1} \forall_{g \in G} \; \pi(g)\xi_i - \xi_i \longrightarrow 0$$

(uniformly on compact subsets)

- is **mixing** if

$$\forall_{\xi, \eta \in \mathsf{H}} \; \langle \xi, \pi(\cdot)\eta \rangle \in C_0(G);$$

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General notations

$\mathbb{G}$ – a locally compact quantum group à la Kustermans-Vaes

$L^\infty(\mathbb{G})$ – a von Neumann algebra equipped with the *coproduct*

$$\Delta : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G})$$

carrying all the information about $\mathbb{G}$

$\mathrm{C}_0(\mathbb{G})$ – the corresponding (reduced) C^*-object, $\mathrm{C}_b(\mathbb{G}) := \mathrm{M}(\mathrm{C}_0(\mathbb{G}))$

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$L^2(\mathbb{G})$ – the GNS Hilbert space of the *right invariant Haar weight* on $\mathbb{G}$

$W^{\mathbb{G}} \in B(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$ – the multiplicative unitary associated to $\mathbb{G}$:

$$\Delta(f) = W^{\mathbb{G}}(f \otimes 1)(W^{\mathbb{G}})^*, \quad f \in L^\infty(\mathbb{G}).$$

Note the inclusions

$$\mathrm{C}_0(\mathbb{G}) \subset C_b(\mathbb{G}) \subset L^\infty(\mathbb{G}) = \mathrm{C}_0(\mathbb{G})''$$

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Each LCQG $\mathbb{G}$ admits the dual LCQG $\widehat{\mathbb{G}}$.

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In particular for G – locally compact group

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Further properties of LCQGs

Definition

A locally compact quantum groups $\mathbb{G}$ is

- **compact** if $C_0(\mathbb{G})$ is unital (equivalently the Haar weights are finite);
- **discrete** if $\hat{\mathbb{G}}$ is compact;
- **unimodular** if the left and right Haar weights coincide;
- **of Kac type** if the so-called scaling group is trivial (the antipode is a bounded map);
- **amenable** if $L^\infty(\mathbb{G})$ admits a bi-invariant mean;
- **coamenable** if the universal and reduced algebras $C_0(\mathbb{G})$ and $C_0^u(\mathbb{G})$ are naturally isomorphic;
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Some examples of locally compact quantum groups

- locally compact groups (all coamenable);
- duals of locally compact groups (all amenable);
- quantum deformations of classical Lie groups: for example $SU_q(2)$, quantum $ax+b$, $E_q(2)$ (amenable and coamenable, usually not Kac);
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The operators $(\iota \otimes \omega_{\xi,\eta})(U) \in \mathrm{C}_b(\mathbb{G})$, where $\xi, \eta \in \mathrm{H}$, are called **coefficients** of U.

Representations of $\mathbb{G}$ are in a 1-1 correpondence with C^*-representations of $\mathrm{C}_0^u(\widehat{\mathbb{G}})$.

One can also **tensor** representations of $\mathbb{G}$ ($U \mathbin{\bigcirc\!\!\!\!\top} V$), take **direct sums** ($U \oplus V$) and pass to a **contragredient representation** U^c.

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Representations of LCQGs – continued

Definition

A representation U of $\mathbb{G}$ is *mixing* if all its coefficients belong to $C_0(\mathbb{G})$. It *has almost invariant vectors* if there exists a net of unit vectors $(\xi_i)_{i\in I}$ such that for all $a \in \mathrm{C}_0(\mathbb{G})$

$$U(a \otimes \xi_i) - a \otimes \xi_i \longrightarrow 0$$

– equivalently for all $b \in \mathrm{C}_0^u(\widehat{\mathbb{G}})$

$$\phi_U(b)\xi_i - \hat{\epsilon}(b)\xi_i \longrightarrow 0 \quad \text{strictly.}$$

The multiplicative unitary $W^{\mathbb{G}}$ plays the role of the left regular representation of $\mathbb{G}$ on $L^2(\mathbb{G})$; it is mixing.

Definitions and first facts

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Space of representations

$\mathbb{G}$ – second countable locally compact quantum group, H – fixed infinite dimensional separable Hilbert space. Then $\mathrm{Rep}_{\mathbb{G}}(\mathsf{H})$ is a Polish space with a natural ('point-weak') topology.
It is equipped with two natural operations: **direct sum** (finite or countable) and **tensoring** (after we fix some unitary identifications of $\mathsf{H} \otimes \mathsf{H}$ with H, etc.).

Lemma (DFSW)

Suppose $\mathcal{R} \subset \mathrm{Rep}_{\mathbb{G}}(\mathsf{H})$

1. is stable under unitary equivalence;
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HAP and density of mixing representations

Theorem (DFSW)

A second countable locally compact quantum group $\mathbb{G}$ has HAP if and only if the set of mixing representations is dense in $\mathrm{Rep}_{\mathbb{G}}(\mathsf{H})$.

Proof.

$\Longleftarrow$

Approximate the trivial representation with mixing ones and take their direct sum (still mixing!).

$\Longrightarrow$

Use the lemma with $\mathcal{R}$ – mixing representations. □

Ideas go back to the work of Halmos for $\mathbb{Z}$.

(T) via ‘typical’ representations

Theorem (Kerr-Pichot, 2012)

Let G – classical locally compact group, second countable. Then G does not have Property (T) if and only if weakly mixing representations are dense in $\mathrm{Rep}_G(\mathsf{H})$.

We want to show the same for (a class of) quantum groups.

Recall: $U \in \mathrm{Rep}_{\mathbb{G}}(\mathsf{H})$ is weakly mixing if $U \textcircled{\top} U^c$ is ergodic.

- when is the class of weakly mixing representations stable under tensoring?
- when not (T) means that there is a weakly mixing representation with almost invariant vectors?

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Weak mixing representations revisited

Classically: U is not weakly mixing if and only if $U ⓣ U^c$ contains an invariant vector if and only if U contains a finite dimensional subrepresentation.
More generally: $U ⓣ V^c$ contains a fixed vector if and only if U and V contain the same finite dimensional subrepresentation.

Lemma (Chen+Ng 2015, see also Kyed+Sołtan, Viselter)

If $\mathbb{G}$ is of Kac type, then a representation of $\mathbb{G}$ is weakly mixing if and only if it does not contain a finite dimensional subrepresentation; hence then the class of weakly mixing representations is stable under tensoring.

Weak mixing representations with almost invariant vectors

Let us contradict the statement: there is a representation of $\mathbb{G}$ which is weakly mixing and has almost invariant vectors.

Definition

$\mathbb{G}$ has Property $(\mathrm{T})^{1,1}$ (of Bekka and Valette) if for every representation U of $\mathbb{G}$ with almost invariant vectors $U \textcircled{\scriptsize T} U^c$ has a fixed vector.

Obviously $(\mathrm{T}) \Longrightarrow (\mathrm{T})^{1,1}$.

Theorem (Bekka and Valette)

For classical groups $(\mathrm{T}) \Longleftrightarrow (\mathrm{T})^{1,1}$.

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Discrete quantum groups with low duals

$\mathbb{\Gamma}$ – discrete quantum group. Then

$$c_0(\mathbb{\Gamma}) = \bigoplus_{i \in I} M_{n_i}$$

Definition

$\mathbb{\Gamma}$ as above has a low dual if $\sup_{i \in I} n_i < \infty$.

In other words, we have a uniform bound on the size of irreducible representations of the compact quantum group dual to $\mathbb{\Gamma}$.

Main theorem

Theorem (DSV, 2016)

Let $\mathbb{\Gamma}$ – discrete unimodular second countable quantum group with a low dual. Then $\mathbb{\Gamma}$ has Property (T) if and only if it has Property $(T)^{1,1}$

Corollary (DSV, 2016)

Let $\mathbb{\Gamma}$ – discrete unimodular second countable quantum group with a low dual. Then $\mathbb{\Gamma}$ does not have (T) if and only if weakly mixing representations form a dense G_δ-set in $\mathrm{Rep}_{\mathbb{\Gamma}}(\mathsf{H})$.

Main theorem – ingredients of the proof

Problem: assume $\mathbb{\Gamma}$ does not have (T). Construct a representation U of $\mathbb{\Gamma}$ with almost invariant vectors such that $U \textcircled{\top} U^c$ contains no invariant vector.

Idea (Jolissaint): use non (T) of $\mathbb{\Gamma}$ to construct a semigroup of states on the algebra $C(\widehat{\mathbb{\Gamma}})$ with particular properties

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Actual ingredients:

- development of the notion of Kazhdan pairs for quantum groups with Property (T);
- application of this to showing that if a locally compact $\mathbb{G}$ is of Kac type and does not have (T) then one can find a net of positive-definite normalised positive elements in $C_b(\mathbb{G})$ which converge to 1 strictly, but not in norm;
- use of Yukio Arano's work on central Property (T) to show that for discrete unimodular case one can choose the elements above in the centre;
- construction of a strongly unbounded symmetric generating functional L on $\widehat{\mathbb{\Gamma}}$;
- using L to generate a convolution semigroup of states μ_t on $C^u(\widehat{\mathbb{\Gamma}})$;
- building out of μ_t 'symmetric' GNS representations of $C^u(\widehat{\mathbb{\Gamma}})$ (thus self-contragredient representations U_t of $\mathbb{\Gamma}$);
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Other consequences of (T)$\Longleftrightarrow$(T)^{1,1}

Theorem (DSV, 2016)

Let $\mathbb{\Gamma}$ – discrete unimodular second countable quantum group with a low dual. Then the following are equivalent:

1. $\mathbb{\Gamma}$ has Property (T);
2. for every action of $\mathbb{\Gamma}$ on a von Neumann algebra invariant states are limits of *normal* invariant states;
3. every ergodic action of $\mathbb{\Gamma}$ on a von Neumann algebra preserving a faithful normal state is strongly operator ergodic (i.e. asymptotically invariant nets of elements in the von Neumann algebra are trivial).

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References

M.Daws, P.Fima, A.S. and S.White, Haagerup property for locally compact quantum groups, Crelle, 2016.

M.Daws, A.S. and A.Viselter, Around Property (T) for locally compact quantum groups, preprint 2016.