Weak Amenability of Central Beurling Algebras

Groups and Operators, Gothenburg, August 16, 2016

Yong Zhang

Department of Mathematics
University of Manitoba, Winnipeg, Canada

Based on Joint work with V. Shepelska

Let G be a locally compact group.
A Haar-measurable function ω: $G \to \mathbb{R}^+$ is a weight if it satisfies

$$\omega(xy) \leq \omega(x)\omega(y) \quad (x, y \in G).$$

Let G be a locally compact group.
A Haar-measurable function $\omega\colon G \to \mathbb{R}^+$ is a weight if it satisfies

$$\omega(xy) \le \omega(x)\omega(y) \quad (x, y \in G).$$

Let ω be a weight on G. We consider the space

$$L^1(G, \omega) = \{f : \ f\omega \in L^1(G)\}.$$

With the convolution product and the norm

$$\|f\|_\omega = \int_G |f(t)|\omega(t)dt \quad (f \in L^1(G, \omega))$$

$L^1(G, \omega)$ is a Banach algebra, called a Beurling algebra on G.

Let G be a locally compact group.
A Haar-measurable function ω: $G \to \mathbb{R}^+$ is a weight if it satisfies

$$\omega(xy) \leq \omega(x)\omega(y) \quad (x, y \in G).$$

Let ω be a weight on G. We consider the space

$$L^1(G, \omega) = \{f : f\omega \in L^1(G)\}.$$

With the convolution product and the norm

$$\|f\|_\omega = \int_G |f(t)|\omega(t)dt \quad (f \in L^1(G, \omega))$$

$L^1(G, \omega)$ is a Banach algebra, called a Beurling algebra on G.
Given a weight ω there is always a continuous weight $\tilde{\omega}$ such that

$$L^1(G, \omega) \simeq L^1(G, \tilde{\omega}).$$

So, in the sequel, we always assume ω be continuous.

central Beurling algebras

The center algebra of a $L^1(G, \omega)$ is called a central Beurling algebra. We denote it by $ZL^1(G, \omega)$.

central Beurling algebras

The center algebra of a $L^1(G,\omega)$ is called a central Beurling algebra. We denote it by $ZL^1(G,\omega)$.

- $ZL^1(G,\omega) \neq \{0\}$ iff G is an [IN] group (i.e. a group with a compact neighborhood of e that is invariant under inner automorphisms);

central Beurling algebras

The center algebra of a $L^1(G,\omega)$ is called a central Beurling algebra. We denote it by $ZL^1(G,\omega)$.

- $ZL^1(G,\omega) \neq \{0\}$ iff G is an [IN] group (i.e. a group with a compact neighborhood of e that is invariant under inner automorphisms);
- Denote by $I(G)$ the set of all inner automorphisms of G. Then

  $$f \in ZL^1(G,\omega) \quad \text{iff} \quad f \in L^1(G,\omega) \text{ and } f \circ \beta = f \text{ for all } \beta \in I(G))$$

  which means $f(gxg^{-1}) = f(x)$ $(g, x \in G)$.

We study weak amenability of $ZL^1(G, \omega)$.

We study weak amenability of $ZL^1(G,\omega)$.

- We note $ZL^1(G,\omega) = L^1(G,\omega)$ if G is commutative, and $L^1(G,\omega)$ is always weakly amenable if $\omega = 1$, (B. E. Johnson 1991).

We study weak amenability of $ZL^1(G,\omega)$.

- We note $ZL^1(G,\omega) = L^1(G,\omega)$ if G is commutative, and $L^1(G,\omega)$ is always weakly amenable if $\omega = 1$, (B. E. Johnson 1991).

  This is no longer true if ω is not constant. In fact

- (N. Gronbaek 1989, Y. Z. 2014) for commutative G, $L^1(G,\omega)$ is weakly amenable if and only if there is no non-trivial continuous group homomorphism Φ: $G \to (\mathbb{C}, +)$ such that

$$\sup_{g\in G} \frac{|\Phi(g)|}{\omega(g)\omega(g^{-1})} < \infty.$$

For $x \in G$ the conjugacy class of x is the set

$$C_x = \{\beta(x) : \beta \in I(G)\} = \{gxg^{-1} : g \in G\}.$$

For $x \in G$ the conjugacy class of x is the set

$$C_x = \{\beta(x) : \beta \in I(G)\} = \{gxg^{-1} : g \in G\}.$$

The group G is a $[FC]^-$ group if C_x is precompact for each $x \in G$.

For $x \in G$ the conjugacy class of x is the set

$$C_x = \{\beta(x) : \beta \in I(G)\} = \{gxg^{-1} : g \in G\}.$$

The group G is a $[FC]^-$ group if C_x is precompact for each $x \in G$.

The good thing for us is that, to study the central Beurling algebra, one may always assume that $G \in [FC]^-$ from the following result due to Liukkonen and Mosak (1977).

- There is an open normal subgroup G_0 of G that belongs to $[FC]^-$ such that

$$ZL^1(G, \omega) \simeq ZL^1(G_0, \omega).$$

For $x \in G$ the conjugacy class of x is the set

$$C_x = \{\beta(x) : \beta \in I(G)\} = \{gxg^{-1} : g \in G\}.$$

The group G is a $[FC]^-$ group if C_x is precompact for each $x \in G$.

The good thing for us is that, to study the central Beurling algebra, one may always assume that $G \in [FC]^-$ from the following result due to Liukkonen and Mosak (1977).

- There is an open normal subgroup G_0 of G that belongs to $[FC]^-$ such that
$$ZL^1(G, \omega) \simeq ZL^1(G_0, \omega).$$
- We note that $[FC]^-$ groups are all amenable $[IN]$ groups (Leptin 1968).

compactly generated groups

Let G be a compactly generated locally compact group. Then there is a symmetrical, open and precompact neighborhood U of the unit e in G such that $G = \cup_{n=1}^{\infty} U^n$.

compactly generated groups

Let G be a compactly generated locally compact group. Then there is a symmetrical, open and precompact neighborhood U of the unit e in G such that $G = \cup_{n=1}^{\infty} U^n$. This allows us to define a length function on G:

$$|x| = \min\{n \in \mathbb{N} : x \in U^n\}.$$

For each $\alpha \geq 0$ this length associates a weight on G in the form

$$\omega_\alpha(x) = (1 + |x|)^\alpha \quad (x \in G).$$

We call it a polynomial weight on G. This is indeed a symmetric upper semicontinuous weight.

compactly generated groups

Let G be a compactly generated locally compact group. Then there is a symmetrical, open and precompact neighborhood U of the unit e in G such that $G = \cup_{n=1}^{\infty} U^n$. This allows us to define a length function on G:

$$|x| = \min\{n \in \mathbb{N} : x \in U^n\}.$$

For each $\alpha \geq 0$ this length associates a weight on G in the form

$$\omega_\alpha(x) = (1 + |x|)^\alpha \quad (x \in G).$$

We call it a polynomial weight on G. This is indeed a symmetric upper semicontinuous weight.

The following is our main result.

Theorem 1

Let G be a compactly generated (non-compact) $[FC]^-$ group and ω_α be a polynomial weight on G. Then $ZL^1(G, \omega_\alpha)$ is weakly amenable if and only if $\alpha < 1/2$.

Special cases of the theorem for commutative groups are as follows.

Special cases of the theorem for commutative groups are as follows.

- For $G = (\mathbb{Z}, +)$,

Special cases of the theorem for commutative groups are as follows.

- For $G = (\mathbb{Z}, +)$, take $|x|$ simply the absolute value of $x \in \mathbb{Z}$.

Special cases of the theorem for commutative groups are as follows.

- For $G = (\mathbb{Z}, +)$, take $|x|$ simply the absolute value of $x \in \mathbb{Z}$. Then

$$ZL^1(\mathbb{Z}, \omega_\alpha) = L^1(\mathbb{Z}, \omega_\alpha)$$

is weakly amenable iff $\alpha < 1/2$. (Bade-Curtis-Dales 1987).

Special cases of the theorem for commutative groups are as follows.

- For $G = (\mathbb{Z}, +)$, take $|x|$ simply the absolute value of $x \in \mathbb{Z}$. Then

$$ZL^1(\mathbb{Z}, \omega_\alpha) = L^1(\mathbb{Z}, \omega_\alpha)$$

  is weakly amenable iff $\alpha < 1/2$. (Bade-Curtis-Dales 1987).

- For $G = (\mathbb{R}^n, +)$

Special cases of the theorem for commutative groups are as follows.

- For $G = (\mathbb{Z}, +)$, take $|x|$ simply the absolute value of $x \in \mathbb{Z}$. Then

$$ZL^1(\mathbb{Z}, \omega_\alpha) = L^1(\mathbb{Z}, \omega_\alpha)$$

is weakly amenable iff $\alpha < 1/2$. (Bade-Curtis-Dales 1987).

- For $G = (\mathbb{R}^n, +)$ we may take

$$|x| = \max_{1 \le i \le n} |x_i|$$

(or equivalently $|x| = (\sum_{i=1}^n |x_i|^2)^{\frac{1}{2}}$) for $x = (x_1, x_2, \ldots x_n) \in \mathbb{R}^n$.

Special cases of the theorem for commutative groups are as follows.

- For $G = (\mathbb{Z}, +)$, take $|x|$ simply the absolute value of $x \in \mathbb{Z}$. Then

  $$ZL^1(\mathbb{Z}, \omega_\alpha) = L^1(\mathbb{Z}, \omega_\alpha)$$

  is weakly amenable iff $\alpha < 1/2$. (Bade-Curtis-Dales 1987).

- For $G = (\mathbb{R}^n, +)$ we may take

  $$|x| = \max_{1 \le i \le n} |x_i|$$

  (or equivalently $|x| = (\sum_{i=1}^n |x_i|^2)^{\frac{1}{2}}$) for $x = (x_1, x_2, \ldots x_n) \in \mathbb{R}^n$. Then

  $$ZL^1(\mathbb{R}^n, \omega_\alpha) = L^1(\mathbb{R}^n, \omega_\alpha)$$

  is weakly amenable iff $\alpha < 1/2$.

weak amenability

Let $\mathcal{A}$ be a Banach algebra and X a Banach $\mathcal{A}$-bimodule. A linear map $D\colon \mathcal{A} \to X$ is a derivation if it satisfies

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in \mathcal{A}).$$

weak amenability

Let $\mathcal{A}$ be a Banach algebra and X a Banach $\mathcal{A}$-bimodule. A linear map $D\colon \mathcal{A} \to X$ is a derivation if it satisfies

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in \mathcal{A}).$$

Suppose that $\mathcal{A}$ is a commutative Banach algebra. An $\mathcal{A}$-module X is called symmetric if $a \cdot x = x \cdot a$ for all $x \in X$ and $a \in \mathcal{A}$. For example, $\mathcal{A}^*$ is a symmetric Banach $\mathcal{A}$-module.

weak amenability

Let $\mathcal{A}$ be a Banach algebra and X a Banach $\mathcal{A}$-bimodule. A linear map $D\colon \mathcal{A} \to X$ is a derivation if it satisfies

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in \mathcal{A}).$$

Suppose that $\mathcal{A}$ is a commutative Banach algebra. An $\mathcal{A}$-module X is called symmetric if $a \cdot x = x \cdot a$ for all $x \in X$ and $a \in \mathcal{A}$. For example, $\mathcal{A}^*$ is a symmetric Banach $\mathcal{A}$-module.

Definition 1

A commutative Banach algebra $\mathcal{A}$ is weakly amenable if there is no non-zero continuous derivation $D\colon \mathcal{A} \to X$ for all symmetric Banach $\mathcal{A}$-modules X.

weak amenability

Let $\mathcal{A}$ be a Banach algebra and X a Banach $\mathcal{A}$-bimodule. A linear map $D\colon \mathcal{A} \to X$ is a derivation if it satisfies

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in \mathcal{A}).$$

Suppose that $\mathcal{A}$ is a commutative Banach algebra. An $\mathcal{A}$-module X is called symmetric if $a \cdot x = x \cdot a$ for all $x \in X$ and $a \in \mathcal{A}$. For example, $\mathcal{A}^*$ is a symmetric Banach $\mathcal{A}$-module.

Definition 1

A commutative Banach algebra $\mathcal{A}$ is weakly amenable if there is no non-zero continuous derivation $D\colon \mathcal{A} \to X$ for all symmetric Banach $\mathcal{A}$-modules X.

Theorem (Bade-Curtis-Dales 1987)

The commutative Banach algebra $\mathcal{A}$ is weakly amenable if and only if there is no non-zero continuous derivation $D\colon \mathcal{A} \to \mathcal{A}^*$.

a necessity result

Proposition 2

Let $G \in [FC]^-$ and ω be a weight on G. Suppose that there exists a non-trivial continuous group homomorphism $\Phi : G \to (\mathbb{C}, +)$ such that

$$\sup_{t \in G} \frac{|\Phi(t)|}{\omega(t)\omega(t^{-1})} < \infty.$$

Then $ZL^1(G, \omega)$ is not weakly amenable.

a necessity result

Proposition 2

Let $G \in [FC]^-$ and ω be a weight on G. Suppose that there exists a non-trivial continuous group homomorphism $\Phi : G \to (\mathbb{C}, +)$ such that

$$\sup_{t \in G} \frac{|\Phi(t)|}{\omega(t)\omega(t^{-1})} < \infty.$$

Then $ZL^1(G, \omega)$ is not weakly amenable.

proof.

There exists a conjugate invariant, precompact and open subset U of G such that Φ is nontrivial on U. Define

$$D(h)(t) = \int_U \Phi(t^{-1}\xi)h(t^{-1}\xi)\, d\xi \quad (t \in G,\ h \in ZL^1(G, \omega)).$$

Then D is indeed a non-trivial continuous derivation from $ZL^1(G, \omega)$ into the symmetrical $ZL^1(G, \omega)$-module $L^\infty(G, 1/\omega)$. □

$[FD]^-$ groups

A locally compact group G is an $[FD]^-$ group if the commutator subgroup of G is precompact. This is equivalent to say that there is a compact normal subgroup K such that G/K is commutative.

$[FD]^-$ groups

A locally compact group G is an $[FD]^-$ group if the commutator subgroup of G is precompact. This is equivalent to say that there is a compact normal subgroup K such that G/K is commutative.

- Clearly $[FD]^- \subset [FC]^-$;

$[FD]^-$ groups

A locally compact group G is an $[FD]^-$ group if the commutator subgroup of G is precompact. This is equivalent to say that there is a compact normal subgroup K such that G/K is commutative.

- Clearly $[FD]^- \subset [FC]^-$;
- Any compactly generated $[FC]^-$ group is an $[FD]^-$ group. (Grosser-Moskowitz 1967)

$[FD]^-$ groups

A locally compact group G is an $[FD]^-$ group if the commutator subgroup of G is precompact. This is equivalent to say that there is a compact normal subgroup K such that G/K is commutative.

- Clearly $[FD]^- \subset [FC]^-$;
- Any compactly generated $[FC]^-$ group is an $[FD]^-$ group. (Grosser-Moskowitz 1967)

Given a weight ω on the $[FD]^-$ group G, it induces a weight $\hat{\omega}$ on G/K that is defined by

$$\hat{\omega}([x]) = \inf_{k \in K} \omega(xk) \quad (x \in G).$$

$[FD]^-$ groups

A locally compact group G is an $[FD]^-$ group if the commutator subgroup of G is precompact. This is equivalent to say that there is a compact normal subgroup K such that G/K is commutative.

- Clearly $[FD]^- \subset [FC]^-$;
- Any compactly generated $[FC]^-$ group is an $[FD]^-$ group. (Grosser-Moskowitz 1967)

Given a weight ω on the $[FD]^-$ group G, it induces a weight $\hat{\omega}$ on G/K that is defined by

$$\hat{\omega}([x]) = \inf_{k \in K} \omega(xk) \quad (x \in G).$$

One sees easily that there are $c_1, c_2 > 0$ such that

$$c_1 \hat{\omega}([x]) \leq \omega(x) \leq c_2 \hat{\omega}([x]) \quad (x \in G).$$

proof to the necessity part of Theorem 1

Take a compact normal subgroup K such that G/K is commutative.

proof to the necessity part of Theorem 1

Take a compact normal subgroup K such that G/K is commutative.

From the structure theorem for compact generated commutative groups, Either $(\mathbb{Z}, +)$ or $(\mathbb{R}, +)$ is a quotient group of G/K.

proof to the necessity part of Theorem 1

Take a compact normal subgroup K such that G/K is commutative.

From the structure theorem for compact generated commutative groups, Either $(\mathbb{Z}, +)$ or $(\mathbb{R}, +)$ is a quotient group of G/K. The quotient map q forms a non-trivial group homomorphism from G/K into $(\mathbb{C}, +)$.

proof to the necessity part of Theorem 1

Take a compact normal subgroup K such that G/K is commutative.

From the structure theorem for compact generated commutative groups, Either $(\mathbb{Z}, +)$ or $(\mathbb{R}, +)$ is a quotient group of G/K. The quotient map q forms a non-trivial group homomorphism from G/K into $(\mathbb{C}, +)$.

Extend q to a continuous group homomorphism Φ: $G \to (\mathbb{C}, +)$.

proof to the necessity part of Theorem 1

Take a compact normal subgroup K such that G/K is commutative.

From the structure theorem for compact generated commutative groups, Either $(\mathbb{Z}, +)$ or $(\mathbb{R}, +)$ is a quotient group of G/K. The quotient map q forms a non-trivial group homomorphism from G/K into $(\mathbb{C}, +)$.

Extend q to a continuous group homomorphism Φ: $G \to (\mathbb{C}, +)$.

If $\alpha \geq 1/2$, then

$$\sup_{t \in G} \frac{|\Phi(t)|}{\omega_\alpha(t)\omega_\alpha(t^{-1})} \leq \sup_{t \in G} \frac{C|t|}{(1+|t|)^{2\alpha}} < \infty,$$

where $C > 0$ is a constant.

proof to the necessity part of Theorem 1

Take a compact normal subgroup K such that G/K is commutative.

From the structure theorem for compact generated commutative groups, Either $(\mathbb{Z}, +)$ or $(\mathbb{R}, +)$ is a quotient group of G/K. The quotient map q forms a non-trivial group homomorphism from G/K into $(\mathbb{C}, +)$.

Extend q to a continuous group homomorphism Φ: $G \to (\mathbb{C}, +)$.

If $\alpha \geq 1/2$, then

$$\sup_{t \in G} \frac{|\Phi(t)|}{\omega_\alpha(t)\omega_\alpha(t^{-1})} \leq \sup_{t \in G} \frac{C|t|}{(1+|t|)^{2\alpha}} < \infty,$$

where $C > 0$ is a constant. Therefore, $L^1(G, \omega_\alpha)$ is not weakly amenable by Proposition 2.

□

a sufficiency result

Proposition 3

Let G be an [FD]$^-$ group and $\omega \geq 1$ be a weight on G satisfying

$$\sup_{n \in \mathbb{N}} \frac{n}{\omega(x^n)\omega(x^{-n})} = \infty \quad (x \in G). \tag{1}$$

Then $ZL^1(G, \omega)$ is weakly amenable.

a sufficiency result

Proposition 3

Let G be an [FD]$^-$ group and $\omega \geq 1$ *be a weight on G satisfying*

$$\sup_{n\in\mathbb{N}} \frac{n}{\omega(x^n)\omega(x^{-n})} = \infty \quad (x \in G). \tag{1}$$

Then $ZL^1(G,\omega)$ *is weakly amenable.*

proof.

Let S be a subgroup of G. Then $L^1(S/K,\hat{\omega})$ is an abelian Buerling algebra.

a sufficiency result

Proposition 3

Let G be an [FD]$^-$ group and $\omega \geq 1$ be a weight on G satisfying

$$\sup_{n\in\mathbb{N}} \frac{n}{\omega(x^n)\omega(x^{-n})} = \infty \quad (x \in G). \tag{1}$$

Then $ZL^1(G,\omega)$ is weakly amenable.

proof.

Let S be a subgroup of G. Then $L^1(S/K,\hat{\omega})$ is an abelian Buerling algebra. Equation (1) ensures that any non-trivial continuous group homomorphism $\Phi : S/K \to \mathbb{C}$ satisfies

$$\sup_{t\in S/K} \frac{|\Phi(t)|}{\hat{\omega}(t)\hat{\omega}(t^{-1})} = \infty.$$

a sufficiency result

Proposition 3

Let G be an [FD]$^-$ group and $\omega \geq 1$ be a weight on G satisfying

$$\sup_{n\in\mathbb{N}} \frac{n}{\omega(x^n)\omega(x^{-n})} = \infty \quad (x \in G). \tag{1}$$

Then $ZL^1(G,\omega)$ is weakly amenable.

proof.

Let S be a subgroup of G. Then $L^1(S/K,\hat{\omega})$ is an abelian Buerling algebra. Equation (1) ensures that any non-trivial continuous group homomorphism $\Phi : S/K \to \mathbb{C}$ satisfies

$$\sup_{t\in S/K} \frac{|\Phi(t)|}{\hat{\omega}(t)\hat{\omega}(t^{-1})} = \infty.$$

So the commutative Beurling algebra $L^1(S/K,\hat{\omega})$ is weakly amenable. This is true for every subgroup S of G.

On the other hand, Equation (1) implies that $\lim_{n\to\infty}(\omega(x^n))^{1/n} = 1$ for every $x \in G$.

On the other hand, Equation (1) implies that $\lim_{n\to\infty}(\omega(x^n))^{1/n} = 1$ for every $x \in G$. We then can apply the following structure theorem of $ZL^1(G, \omega)$.

Lemma 4 (Mosak-Rochester 1978)

Let G be an $[FD]^-$ group with a compact normal subgroup K such that G/K is Abelian. Let $\omega \geq 1$ be a weight on G satisfying

$$\lim_{n\to\infty}(\omega(x^n))^{1/n} = 1 \quad (x \in G).$$

On the other hand, Equation (1) implies that $\lim_{n\to\infty}(\omega(x^n))^{1/n} = 1$ for every $x \in G$. We then can apply the following structure theorem of $ZL^1(G,\omega)$.

Lemma 4 (Mosak-Rochester 1978)

Let G be an $[FD]^-$ group with a compact normal subgroup K such that G/K is Abelian. Let $\omega \geq 1$ be a weight on G satisfying

$$\lim_{n\to\infty}(\omega(x^n))^{1/n} = 1 \quad (x \in G).$$

Then $ZL^1(G,\omega) = \overline{\mathrm{lin}}\{J_\gamma\}_{\gamma\in\Gamma}$, where each J_γ is a complemented ideal of $ZL^1(G,\omega)$ isomorphic to a Beurling algebra of the form $L^1(S_\gamma/K,\hat{\omega})$ for some open normal subgroup S_γ of G.

On the other hand, Equation (1) implies that $\lim_{n\to\infty}(\omega(x^n))^{1/n} = 1$ for every $x \in G$. We then can apply the following structure theorem of $ZL^1(G,\omega)$.

Lemma 4 (Mosak-Rochester 1978)

Let G be an $[FD]^-$ group with a compact normal subgroup K such that G/K is Abelian. Let $\omega \geq 1$ be a weight on G satisfying

$$\lim_{n\to\infty}(\omega(x^n))^{1/n} = 1 \quad (x \in G).$$

Then $ZL^1(G,\omega) = \overline{\mathrm{lin}}\{J_\gamma\}_{\gamma\in\Gamma}$, where each J_γ is a complemented ideal of $ZL^1(G,\omega)$ isomorphic to a Beurling algebra of the form $L^1(S_\gamma/K,\hat{\omega})$ for some open normal subgroup S_γ of G.

From the previous slide, each J_γ is weakly amenable. Then $ZL^1(G,\omega) = \overline{\mathrm{lin}}\{J_\gamma\}_{\gamma\in\Gamma}$ is weakly amenable.

□

proof to the sufficiency part of Theorem 1

As a compactly generated $[FC]^-$ group, G is a $[FD]^-$ group.

proof to the sufficiency part of Theorem 1

As a compactly generated $[FC]^-$ group, G is a $[FD]^-$ group.

If $\alpha < 1/2$ we have

$$\sup_{n\in\mathbb{N}} \frac{n}{\omega_\alpha(x^n)\omega_\alpha(x^{-n})} \geq \sup_{n\in\mathbb{N}} \frac{n}{(1+n)^{2\alpha}} = \infty \quad (x \in G),$$

since $|x^n| = |x^{-n}| \leq n$.

proof to the sufficiency part of Theorem 1

As a compactly generated $[FC]^-$ group, G is a $[FD]^-$ group.

If $\alpha < 1/2$ we have

$$\sup_{n\in\mathbb{N}} \frac{n}{\omega_\alpha(x^n)\omega_\alpha(x^{-n})} \geq \sup_{n\in\mathbb{N}} \frac{n}{(1+n)^{2\alpha}} = \infty \quad (x \in G),$$

since $|x^n| = |x^{-n}| \leq n$.

Then Proposition 3 ensures that $ZL^1(G, \omega_\alpha)$ is weakly amenable.

□

Thank You!