MATEMATIK
Göteborgs universitet
Tentamen 2016-03-18, 8:30-12:30

MMGF30, Transformteori och analytiska funktioner

- **Examiner:** Mahmood Alaghmandan, tel: 772 53 74, Email: mahala@chalmers.se
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- **Materials allowed:** Nothing. Only a language dictionary is allowed.

- **Evaluation criteria:** The exam has 9 questions with 100 points. You need at least 50 points for *pass* (G), and at least 80 points for *distinguished pass* (VG).
- **Solution:** The solution to the exam will appear on the homepage of the course shortly after the exam.

Note: You must justify all of your answers unless otherwise is instructed!!! Unjustified answers will receive little or no marks. The hints are just hints. You are free to choose any correct approach to answer questions, unless the question specifically asks you to use one. There are 9 questions in this exam. Points of each question is indicated at the end. Many of the questions in the exam have more than one part. Be careful that you answer every part.

Questions

Question 1 (a) Find all the roots of $z^2 + 4z + 8 = 0$. And present them in the Cartesian form $a + ib$ and the polar form $|z|e^{i\theta}$. (5 points)

(b) Decompose the following complex function into its real and imaginary parts. (5 points)

$$f(z) = \frac{z^*}{z-1} \quad z \in \mathbb{C} \setminus \{1\}.$$

(Note: No need to simplify your answer to part (b).)

Question 2 Prove that the complex function $f(z) = e^{4z}$ for all $z \in \mathbb{C}$ is analytic and find its derivatives using the Cauchy-Riemann theorem. (6 points)

(Hint: Recall the definition of e^{ib} for $b \in \mathbb{R}$.)

Question 3 Find the value of the complex integrals. Briefly justify your answer. (5 points each)

(a) $\int_C e^{4z}dz$ for the path C parametrized as $z(t) = t^2 - it$ for the real value t changing from 0 to 2.

(b) $\oint_C \frac{1}{z-1}dz$ where C is the counterclockwise unit circle centred at i with radius 1/2.

(c) $\oint_C (z^* - 1)dz$ where C is the clockwise unit circle centred at 1 with radius 1.

Question 4 (a) Find the Laurent coefficients a_n for $n \leq 0$ of the Laurent series of $f(z) = \frac{e^z \sin(\pi z/2)}{z+1}$ around -1. (7 points)

(Note: We do not need a_n for $n \geq 1$.)

(b) Decide $z_0 = -1$ what kind of singularity for $f(z)$ is. (2 points)

Question 5 Find the value of the integral $\int_0^\infty \frac{\cos(-3x)}{x^2+1}dx$. (12 points)

Question 6 Find the pattern of the Fourier coefficients $\{a_0, a_n, b_n : n \in \mathbb{N}\}$ for the periodic function $f(x)$ which is defined as by $f(x) = |x|$ on $-1 \leq x \leq 1$ and then $f(x+2) = f(x)$ for every $x \in \mathbb{R}$. (14 points)

Question 7 (a) Find the Fourier Transform $\mathcal{F}(f)(\omega)$ of $f(t) = \frac{1}{t^2 + \alpha^2}$ for a real constant $\alpha > 0$. (8 points)

(b) Solve the following integral. (6 points)

$$\int_0^\infty \frac{1}{(t^2+5)^2}dt.$$

(Hint: You may want to use the Plancherel theorem and the answer to part (a).)

Question 8 Prove the following identities.

(a) Prove $\mathcal{F}\big(f'\big)(\omega) = (i\omega)\mathcal{F}\big(f\big)(\omega)$ where for f so that its Fourier transform exists and $\lim_{R\to\infty} f(R) = \lim_{R\to-\infty} f(R) = 0$. (6 point)

(b) Prove $\mathcal{L}\big(e^{at}f(t)\big)(s) = \mathcal{L}\big(f\big)(s-a)$ where for f so that its Laplace transform exists. (4 point)

Question 9 Solve the following initial value problem using the Laplace transform. (10 points)

- $y'' + 4y' + 4y = \cos(-2x)$ with $y(0) = 0$ and $y'(0) = 0$.

I really enjoyed teaching this course and hope it was also fun for you too. Be in touch. Good Luck in this and your other exams!

Bests,
Mahmood

Formula Sheet

Some trigonometric identities

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B),$$
$$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B),$$
$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

$$\sin(2A) = 2\sin(A)\cos(A),$$
$$\cos(2A) = \cos^2(A) - \sin^2(A) = 2\cos^2(A) - 1 = 1 - 2\sin^2(A).$$

$$\cos(A)\cos(B) = \frac{1}{2}(\cos(A+B) + \cos(A-B)),$$

$$\sin(A)\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B)),$$

$$\sin(A)\cos(B) = \frac{1}{2}(\sin(A+B) + \sin(A-B)).$$

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

Cauchy-Riemann conditions

For a complex function $f(x+iy) = u(x,y) + iv(x,y)$, the Cauchy-Riemann conditions are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Laurent series

If $f(z)$ is analytic throughout an annulus D encircling z_0 and C is any closed counterclockwise circle centred at z_0 and lied completely inside D. Then the Laurent series of f around z_0 is defined as follows

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

where

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz.$$

Fourier series

Suppose $f(x)$ is a $2L$-periodic piecewise smooth function, then Fourier series of f is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(n\omega x) + b_n \sin(n\omega x))$$

where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\omega x)dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\omega x)dx$$

for $\omega = \dfrac{2\pi}{2L}$ and $n \in \mathbb{N}$.

Complex Fourier series

Let $f(x)$ be a $2L$-periodic piecewise smooth function. The complex form of the Fourier series of f is

$$\sum_{n=-\infty}^{\infty} c_n e^{in\omega x}$$

where $\omega = 2\pi/(2L)$ and $c_n = \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(x)e^{-in\omega x}dx$.

Fourier transform

Let $f(t)$ and $F(\omega)$ be integrable functions then the Fourier transform of f is

$$\mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

and its inverse Fourier transform of F is

$$\mathcal{F}^{-1}(F)(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$$

Laplace transform

Suppose that $f(t)$ is a function define on $[0, \infty)$, the Laplace transform of f if it exists is defined by

$$\mathcal{L}(f)(s) = \int_{0}^{\infty} f(t)e^{-st}dt.$$

Laplace transform table

The following table has the Laplace transform of the required functions.

$f(t)$	$\mathcal{L}(f(t))(s)$	Limitations	Argument
1	$\frac{1}{s}$	$s > 0$	Example J2
e^{at}	$\frac{1}{s-a}$	$s > a$	Proposition J4
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$	Example J7 (4)
$\cosh(at)$	$\frac{s}{s^2-a^2}$	$s > a$	Proposition J8 (1)
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$s > a$	Proposition J8 (2)
$\cos(at)$	$\frac{s}{s^2+a^2}$	$s > 0$	Proposition J8 (3)
$\sin(at)$	$\frac{a}{s^2+a^2}$	$s > 0$	Proposition J8 (4)
$e^{at} cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$	Assignment 6 Question 1
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$	Assignment 6 Question 1
$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$	$s > 0$	Assignment 6 Question 2
$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	$s > 0$	Assignment 6 Question 2