Q1 (a) $z^2+4z+8=0$

$$z = \frac{-4 \pm \sqrt{16-32}}{2} = \frac{-4 \pm \sqrt{-16}}{2} = -2 \pm i2$$

$z_1 = -2+i2 \qquad z_2 = -2-i2$

$|z_1| = |z_2| = \sqrt{4+4} = 2\sqrt{2}$

$\theta_{z_1} = \frac{3\pi}{4}, \quad \theta_{z_2} = \frac{5\pi}{4}$

$z_1 = 2\sqrt{2}\, e^{i\frac{3\pi}{4}} \qquad z_2 = 2\sqrt{2}\, e^{i\frac{5\pi}{4}}$

(b)

$$f(z) = f(x+iy) = \frac{(x+iy)^*}{(x+iy)-1} = \frac{x-iy}{(x-1)+iy}$$

$$= \frac{(x-iy)\big((x-1)-iy\big)}{(x-1)^2+y^2} = \frac{x(x-1)-y^2+i\big(-xy-y(x-1)\big)}{(x^{\cancel{2}}-1)^2+y^2}$$

$$= \frac{x(x-1)-y^2}{(x-1)^2+y^2} + i\,\frac{-xy-y(x-1)}{(x-1)^2+y^2}$$

real $\qquad\qquad$ imaginary

$$u(x,y) = \frac{x(x-1)-y^2}{(x-1)^2+y^2}, \quad v(x,y) = \frac{-xy-y(x-1)}{(x-1)^2+y^2}$$

Q2)

$f(z) = e^{4z}$ let $z = x+iy$

$f(x+iy) = e^{4x+i4y} = e^{4x} e^{i4y}$

$= e^{4x}\left(\cos(4y) + i\sin(4y)\right) = \underbrace{e^{4x}\cos(4y)}_{u(x,y)} + i\,\underbrace{e^{4x}\sin(4y)}_{v(x,y)}$

$\frac{\partial u}{\partial x} = 4e^{4x}\cos(4y)$ $\qquad$ $\frac{\partial u}{\partial y} = -4e^{4x}\sin(4y)$

$\frac{\partial v}{\partial x} = 4e^{4x}\sin(4y)$ $\qquad$ $\frac{\partial v}{\partial y} = 4e^{4x}\cos(4y)$

since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$\Rightarrow$ f is always differenciable $\Rightarrow$ f is entire

By Cauchy-Rieman formula:

$f'(z) = f'(x+iy) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 4e^{4x}\cos(4y) + i\,4e^{4x}\sin(4y)$

$= 4e^{4x}\left(\cos(4y) + i\sin(4y)\right) = 4e^{4x}e^{i4y} =$

$= 4e^{4(x+iy)} = 4e^{4z}$

③

(a) Note that $f(z) = e^{4z}$ is analytic and $F(z) =$ $\frac{e^{4z}}{4}$ is its predual. So by FTC we have

$$\int_C e^{4z}\, dz = \left(\frac{e^{4z}}{4}\right]_0^{4-2i} = \frac{e^{4(4-2i)} - 1}{4}$$

$$= \frac{e^{16-8i} - 1}{4}$$

(b) Since $z_0 = 1$ is the only singularity of $\frac{1}{z-1}$ which is outside C, we have

$$\oint_C \frac{1}{z-1}\, dz = 0 \quad \text{by Cauchy Thm.}$$

(c)

$$\oint_C (z^* - 1)\, dz.$$

Note that counterclockwise circel centered at 1 is $z(\theta) = e^{i\theta} + 1$ for $\theta \in [0, 2\pi]$. So we get

$$\int_C (z^*-1)\,dz = -\int_D (z^*-1)\,dz = -\int_0^{2\pi} \left((1+e^{i\theta})^*-1\right) ie^{i\theta}\,d\theta$$

$$= -\int_0^{2\pi} (\cancel{1}+e^{-i\theta}-\cancel{1})\, ie^{i\theta}\,d\theta = -\int_0^{2\pi} i\,d\theta = -2\pi i$$

Q4 By the formula:

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

$$a_n = \frac{1}{2\pi i}\oint_C \frac{e^z \sin\left(\frac{\pi z}{2}\right)}{(z+1)^{n+2}}\,dz$$

If $n+2 \leq 0$ then $\dfrac{e^z \sin\left(\frac{\pi z}{2}\right)}{(z+1)^{n+2}}$ is analytic

everywhere $\Rightarrow a_n = 0$ for $n \leq -2$.

For $n=-1$

$$a_n = \frac{1}{2\pi i}\oint_C \frac{e^z \sin\left(\frac{\pi z}{2}\right)}{(z+1)}\,dz \overset{\text{by Cauchy formula}}{=} e^{-1}\sin\left(\frac{-\pi}{2}\right) = -\frac{1}{e}$$

For $n=0$

$$a_n = \frac{1}{2\pi i}\oint_C \frac{e^z \sin\left(\frac{\pi z}{2}\right)}{(z+1)^2}\,dz = e^{-1}\sin\left(\frac{-\pi}{2}\right) + e^{-1}\frac{\pi}{2}\cancel{\cos\left(\frac{-\pi}{2}\right)}^{\,0}$$

$$= -\frac{1}{e}$$

(b) It is a pole of order 1.

Q5) Note that $\displaystyle\int_0^{+\infty} \frac{\cos(-3x)}{x^2+1}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\cos(-3x)}{x^2+1}\,dx$

$$= \frac{1}{2}\,\mathrm{Re}\left(\int_{-\infty}^{\infty} \frac{e^{-i3x}}{x^2+1}\,dx\right) = \frac{1}{2}\,\mathrm{Re}\left(-2\pi i \operatorname*{Res}_{-i} \frac{e^{-i3x}}{x^2+1}\right)$$

But

$$\operatorname*{Res}_{-i} \frac{e^{-i3x}}{x^2+1} = \frac{1}{2\pi i}\oint_C \frac{e^{-i3x}}{x^2+1}\,dx = \frac{1}{2\pi i}\oint_C \frac{\frac{e^{-i3x}}{x-i}}{x+i}\,dx$$

$$= \frac{e^{-i3(-i)}}{-2i} = \frac{e^{-3}}{-2i}$$

$$\Rightarrow \int_0^{\infty} \frac{\cos(-3x)}{x^2+1}\,dx = \frac{1}{2}\,\mathrm{Re}\left(-2\pi i\,\frac{e^{-3}}{-2i}\right) = \frac{\pi e^{-3x}}{2}.$$

Q6 By the formula for $L=1$ we get

$\Rightarrow w = \frac{2\pi}{2} = \pi$

$$a_0 = \int_{-1}^{1} |x|\,dx = \int_0^1 x\,dx - \int_{-1}^{0} x\,dx = \left(\frac{x^2}{2}\right]_0^1 - \left(\frac{x^2}{2}\right]_{-1}^0$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

Since the function is even $\left(f(-x) = f(x)\right)$ then

all $b_n = 0$

$$a_n = \int_{-1}^{1} |x| \cos(n\pi x)\,dx = \int_0^1 \overbrace{x}^{U} \overbrace{\cos(n\pi x)\,dx}^{dV} -$$

$$\int_{-1}^{0} x\cos(n\pi x)\,dx = \left(+x\frac{\sin(n\pi x)}{n\pi}\right]_0^1 - \int_0^1 \frac{\sin(n\pi x)}{n\pi}\,dx$$

by parts

$$-\left(\frac{x\sin(n\pi x)}{n\pi}\right]_{-1}^{0} + \int_{-1}^{0}\frac{\sin(n\pi x)}{n\pi}\,dx$$

$$= \underbrace{\cancel{\frac{\sin(n\pi)}{n\pi}}}_{0} + \frac{1}{n\pi}\left(\frac{\cos(n\pi x)}{n\pi}\right]_0^1 - \underbrace{\cancel{\frac{\sin(n\pi)}{n\pi}}}_{0} - \frac{1}{n\pi}\left(\frac{\cos(n\pi x)}{n\pi}\right]_{-1}^{0}$$

$$= \frac{1}{n^2\pi^2}\left(\cos(n\pi) - \overset{1}{\cancel{\cos(0)}}\right) - \frac{1}{n^2\pi^2}\left(\overset{1}{\cancel{\cos(0)}} - \overset{\cos(n\pi)}{\cancel{\cos(-n\pi)}}\right)$$

$$= \frac{1}{n^2\pi^2}\left(\cos(n\pi) + \cos(n\pi) - 2\right)$$

$$= \frac{1}{n^2\pi^2}\left(2\cos(n\pi) - 2\right) = \begin{cases} 0 & n \text{ even} \\ \dfrac{-4}{n^2\pi^2} & n \text{ odd} \end{cases}$$

Q7)

$$\mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{t^2+\alpha^2} e^{-i\omega t}\, dt$$

If $\omega < 0 \Rightarrow$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{t^2+\alpha^2}\, dt = \frac{1}{\sqrt{2\pi}} \left(2\pi i \operatorname{Res}_{\alpha i} \frac{e^{-i\omega t}}{\underbrace{t^2+\alpha^2}_{(t+\alpha i)(t-\alpha i)}} \right)$$

$$= \sqrt{2\pi}\, \cancel{i}\, \frac{e^{-i\omega \alpha i}}{2\alpha \cancel{i}} = \frac{\sqrt{2\pi}}{2\alpha} e^{\alpha\omega} = \sqrt{\frac{\pi}{2}}\, \frac{e^{\alpha\omega}}{\alpha}$$

If $\omega > 0 \Rightarrow -\omega < 0$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{t^2+\alpha^2}\, dt = \frac{1}{\sqrt{2\pi}} \left(-2\pi i \operatorname{Res}_{-\alpha i} \frac{e^{-i\omega t}}{(t+\alpha i)(t-\alpha i)} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(-\cancel{2}\pi \cancel{i}\, \frac{e^{-i\omega(-\alpha i)}}{\cancel{-2}\alpha \cancel{i}} \right) = \sqrt{\frac{\pi}{2}}\, \frac{e^{-\omega\alpha}}{\alpha}$$

$$\Rightarrow \mathcal{F}(f)(\omega) = \sqrt{\frac{\pi}{2}}\, \frac{e^{-|\omega|\alpha}}{\alpha}$$

(b)

by Planchard formula

$$\int_0^{\infty} \frac{1}{(t^2+5)^2}\, dt = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{1}{(t^2+5)^2}\, dt =$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} F\left(\frac{1}{t^2+5}\right)^2(\omega)\,d\omega = \frac{1}{2}\int_{-\infty}^{+\infty}\left(\frac{\sqrt{\pi}\,e^{-\sqrt{5}|\omega|}}{\sqrt{2}\,\sqrt{5}}\right)^2 d\omega$$

$$= \frac{\pi}{4\times 5}\int_{-\infty}^{+\infty} e^{-2\sqrt{5}\,|\omega|}\,d\omega$$

$$= \frac{\pi}{20}\int_{-\infty}^{0} e^{2\sqrt{5}\,\omega}\,d\omega + \frac{\pi}{20}\int_{0}^{\infty} e^{-2\sqrt{5}\,\omega}\,d\omega$$

$$= \frac{\pi}{20}\left(\frac{e^{2\sqrt{5}\,\omega}}{2\sqrt{5}}\right]_{-\infty}^{0} + \frac{\pi}{20}\left(\frac{e^{-2\sqrt{5}\,\omega}}{-2\sqrt{5}}\right]_{0}^{\infty}$$

$$= \frac{\pi}{20}\left(\frac{1}{2\sqrt{5}}\right) + \frac{\pi}{20}\left(+\frac{1}{2\sqrt{5}}\right) = \frac{\pi}{20\sqrt{5}}$$

Q8

a)

$$F(f')(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \overbrace{f'(t)}^{dv}\, \overbrace{e^{-i\omega t}}^{u}\, dt = \frac{1}{\sqrt{2\pi}} \left(e^{-i\omega t} f(t) \right]_{-\infty}^{+\infty}$$

$$- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) \left(-i\omega e^{-i\omega t} \right) dt$$

$$= \frac{1}{\sqrt{2\pi}} (0 - 0) + \frac{i\omega}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t}\, dt = (i\omega) F(f)(\omega)$$

(b)

$$\mathcal{L}(e^{at} f(t))(s) = \int_{0}^{\infty} e^{at} f(t)\, e^{-st}\, dt = \int_{0}^{\infty} e^{-(s-a)t} f(t)\, dt$$

$$= F(f)(s-a).$$

Q9

$$\mathcal{L}(y''+4y'+4y) = \mathcal{L}(\cos(-2x))$$

$$\mathcal{L}(y'')+4\mathcal{L}(y')+4\mathcal{L}(y) = \frac{s}{s^2+4}$$

$$s^2\mathcal{L}(y) - s\overset{0}{\cancel{y(0)}} - \overset{0}{\cancel{y'(0)}} + 4\left(s\mathcal{L}(y) - \overset{0}{\cancel{y(0)}}\right) + 4\mathcal{L}(y) = \frac{s}{s^2+4}$$

$$(s^2+4s+4)\mathcal{L}(y) = \frac{s}{s^2+4}$$

$$\mathcal{L}(y) = \frac{s}{(s^2+4)(s+2)^2}$$

$$\frac{s}{(s^2+4)(s+2)^2} = \frac{As+B}{s^2+4} + \frac{Cs+D}{(s+2)^2} =$$

$$= \frac{(As+B)(s^2+4s+4) + (Cs+D)(s^2+4)}{(s^2+4)(s+2)^2}$$

$$= \frac{As^3+4As^2+4As+Bs^2+4Bs+4B+Cs^3+4Cs+Ds^2+4D}{(s^2+4)(s+2)^2}$$

$$\begin{cases} A+C=0 \Rightarrow A=-C \\ 4A+B+D=0 \\ 4A+4B+4C=1 \\ 4B+4D=0 \Rightarrow B=-D \end{cases} \Rightarrow \begin{cases} 4A+B-B=0 \Rightarrow A=0 \Rightarrow C=0 \\ 4B=1 \Rightarrow B=\frac{1}{4} \end{cases}$$

$$D=-\frac{1}{4}$$

$$\mathcal{L}(y) = \frac{\frac{1}{4}}{s^2+4} - \frac{\frac{1}{4}}{(s+2)^2}$$

$$\Rightarrow y = \frac{1}{4}\,\frac{1}{2}\sin(2t) - \frac{1}{4}\,t\,e^{-2t} = \frac{1}{8}\sin(2t) - \frac{1}{4}te^{-2t}.$$