MMGF: Transformteori och analytiska funktioner. VT 2013

Lösningar, 2013-03-15

1. a). funktionen är analytisk för $z \neq 0$ som en produkt av 2 analytiska funktioner

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{1}{2i}\left(e^{x+iy} - e^{-x-iy}\right)$$

$$= \frac{1}{2i}\left(e^{x}(\cos y + i\sin y) - e^{-x}(\cos y - i\sin y)\right)$$

$$= \frac{1}{2}\left(e^{x}\sin y + e^{-x}\sin y\right) + \frac{i}{2}\left(e^{x}\cos y - e^{-x}\cos y\right)$$

$$= \frac{1}{2}(e^{x}+e^{-x})\sin y + \frac{i}{2}(e^{x}-e^{-x})\cos y$$

$$= \cosh x \cdot \sin y + i \sinh x \cos y$$

$$\frac{1}{z^2} = \frac{1}{(x+iy)^2} = \frac{1}{x^2-y^2+2ixy} = \frac{(x^2-y^2)-2ixy}{(x^2-y^2)^2+(2xy)^2} =$$

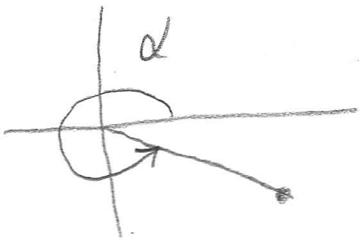
$$\frac{(x^2-y^2)-2ixy}{(x^2+y^2)^2}$$

$$\frac{\sin z}{z^2} = \cosh x \sin y \cdot \frac{x^2-y^2}{(x^2+y^2)^2} - \sinh x \cos y \cdot \frac{-2xy}{(x^2+y^2)^2}$$

$$+ i\left(\cosh x \sin y \cdot \frac{-2xy}{(x^2+y^2)^2} + \sinh x \cos y \cdot \frac{x^2-y^2}{(x^2+y^2)^2}\right)$$

b. $\frac{i-1}{2i-6} = \frac{(i-1)(-2i-6)}{4^2+6^2} = \frac{6+2-6i+2i}{40}$

$= \frac{8-4i}{40} = \frac{2-i}{10}$



$\left|\frac{i-1}{2i-6}\right| = \sqrt{\frac{2^2+1^2}{10^2}}$

$= \sqrt{\frac{5}{100}} = \frac{1}{2\sqrt{5}}$

$\sin\alpha = -\frac{1}{10}$, ligger i IV kvadranten,

$\alpha = 2\pi - \arcsin\frac{1}{10}$ -

$\left|\left(\frac{i-1}{2i-6}\right)^3\right| = \left|\frac{i-1}{2i-6}\right|^3 = \frac{1}{8\cdot 5^{3/2}}$

$\mathrm{Arg}\left(\frac{i-3}{2i-6}\right)^3 = 3\left(2\pi - \arcsin\frac{1}{3}\right) + 2n\pi = 2n\pi - 3\arcsin\frac{1}{3}$

c. $z^2 - 2(i+1)z - \ell + 2i = 0$

$z_{1,2} = i+1 \pm \sqrt{(i+1)^2 - (-\ell+2i)} = i+1 \pm \sqrt{2i+\ell-2i}$

$= i+1 \pm \sqrt{\ell}$

om $\ell \geq 0$, så $z_{1,2} = i+1 \pm \sqrt{\ell} = (1 \pm \sqrt{\ell}) + i$

om $\ell < 0$, så $z_{1,2} = i+1 \pm i\sqrt{-\ell} = 1 + i(1 \pm \sqrt{-\ell})$

d. $\ln(1+i)$: $|1+i| = \sqrt{2}$, $\arg(1+i) = \frac{\pi}{4}$;

$\ln(1+i) = \ln\sqrt{2} + i\frac{\pi}{4}$

$\ln(2+3i)$: $|2+3i| = \sqrt{2^2+3^2} = \sqrt{13}$, $\arg(2+3i) =$ $\arcsin\frac{3}{\sqrt{13}}$, $\ln(2+3i) = \ln\sqrt{13} + i\frac{3}{\sqrt{13}}$

$\sin i = \frac{e^i - e^{-i}}{2} = \frac{\cos 1 + i\sin 1 - \cos(-1) - i\sin(-1)}{2i}$

$= \sin 1$; $|\sin i| = \sin 1$, $\arg \sin i = 0$

2. $f(z) = \dfrac{z}{z^3+1}$; sing punkter : $z^3 = -1$,

$|-1| = 1$; $\arg(-1) = \pi$

$$z_1 = 1 \cdot \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$z_2 = 1 \cdot \left(\cos\left(\frac{\pi}{3} + \frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{3} + \frac{2\pi}{3}\right)\right) = -1$$

$$z_3 = 1 \left(\cos\left(\frac{\pi}{3} + \frac{4\pi}{3}\right) + i\sin\left(\frac{\pi}{3} + \frac{4\pi}{3}\right)\right) = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$$\operatorname*{Res}_{z_1} f(z) = \frac{z_1}{(z-z_1)(z-z_2)(z-z_3)}$$

$$\operatorname*{Res}_{z_1} f(z) = \frac{z_1}{(z_1-z_2)(z_1-z_3)} = \frac{\frac{1}{2} + i\frac{\sqrt{3}}{2}}{\left(\frac{3}{2} + i\frac{\sqrt{3}}{2}\right)\left(i\frac{\sqrt{3}}{2}\right)}$$

$$\operatorname*{Res}_{z_2} f(z) = \frac{z_2}{(z_2-z_1)(z_2-z_3)} ; \quad \operatorname*{Res}_{z_3} f(z) = \frac{z_3}{(z_3-z_1)(z_3-z_2)}$$

$f(z) = \dfrac{z}{e^z-1}$; $e^z - 1 = 0$; $z_n = 2\pi i \cdot n$, $n = 0, \pm 1,$

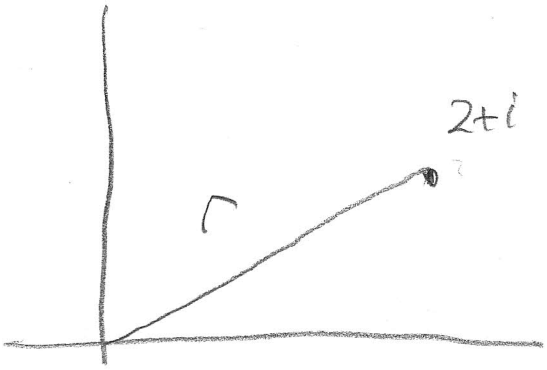
punkten $z=0$ är inte sing, eftersom $\lim\limits_{z\to 0} \dfrac{z}{e^z-1} = 1$

i punkten z_n, $n \neq 0$:

$$\operatorname*{Res}_{z_n} \frac{z}{e^z-1} = \lim_{z\to z_n}\left(\frac{z(z-z_n)}{e^z-1}\right) = \lim_{z\to z_n} \frac{z(z-z_n)}{e^z-e^{z_n}}$$

$$\lim_{z_n} = \frac{z_n}{e^{z_n}} = 2\pi i n.$$

3.



$\Gamma: \; x = 2t,\; y = t \;, \quad 0 \le t \le 1$

$$\int_{0,\,\Gamma}^{2+i} \operatorname{Im}(z)\,dz = \int_0^1 y \frac{dz}{dt}\,dt = \int_0^1 t\,(2+i)\,dt$$

$$= \frac{2+i}{2}$$

Eftersom funktionen inte är analytisk, så beror integralen på vägen.
Funktionen måste vara analytisk, t. ex. e^z, $\frac{1}{1+z}$
funktioner $\frac{1}{z}$, $\frac{1}{z^2}$ passar inte
eftersom 0 är en sing. punkt

4. $$\int_{-\infty}^{\infty} \frac{\sin(3x)}{x^2-2x+8}\,dx = \operatorname{Im}\left(\int_{-\infty}^{\infty} \frac{e^{3x\cdot i}}{x^2-2x+8}\,dx \right)$$

$$= \left(\text{Res. satsen}\right)$$

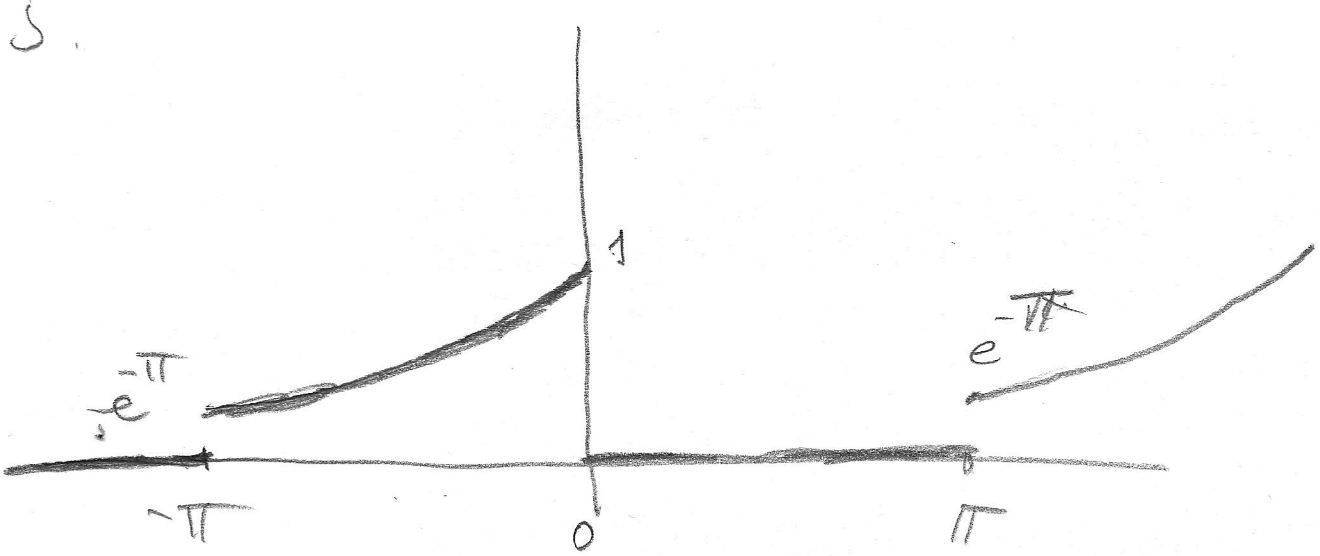
$$\operatorname{Im}\left(2\pi i \sum_{\operatorname{Im} z_n > 0} \operatorname{Res}_{z_n} \frac{e^{3z\cdot i}}{z^2-2z+8} \right)$$

$z^2 - 2z + 8 = 0$; $(z-1)^2 + 7 = 0$; $z+1 = \pm i\sqrt{7}$
$z = 1 \pm i\sqrt{7}$. Polen $z = 1 + i\sqrt{7}$ satisfierar $\operatorname{Im} z > 0$

$$\operatorname{Res}_{1+i\sqrt{7}} \frac{e^{3zi}}{(z-1+i\sqrt{7})(z-1-i\sqrt{7})} =$$

$$= \frac{e^{3i(1+i\sqrt{7})}}{(1+i\sqrt{7} \cancel{-} 1 + i\sqrt{7}} = \frac{e^{3i(1+i\sqrt{7})}}{2i\sqrt{7}}$$

5.



$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} e^{x} e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\cdot\frac{1}{(1-in)}\left(e^{(1-in)x}\Big|_{-\pi}^{0}\right)$$

$$= \frac{1}{2\pi}\cdot\frac{1}{1+in}\left(1 - e^{-\pi(1-in)}\right)$$

$$c_n = \frac{1}{2\pi}\cdot\frac{1}{1+in}\left(1 - e^{-\pi(1-in)}\right)$$

F. serien:

$$\sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\cdot\frac{1}{1+in}\left(1 - e^{-\pi(1-in)}\right) e^{inx}$$

konvergerar: i $x=0$ till $\frac{f(0+0)+f(0-0)}{2}$

$$= \frac{1+0}{2} = \frac{1}{2}$$

i $x = \frac{\pi}{2}$ till 0 (funktionen är kont. i punkten 0

i $x=\pi$ till $\frac{f(\pi+0)+f(\pi-0)}{2} = \frac{e^{-\pi}+0}{2} = \frac{e^{-\pi}}{2}$

6. $4y(x) - y''(x) = 2\delta(x)$

Fouriertransformerar

$$Y(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y(x) e^{-i\omega x} dx$$

$$4Y(\omega) - (i\omega)^2 Y(\omega) = \frac{2}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}$$

$$Y(\omega) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\omega^2 + 4}$$

$$y(x) = \mathcal{F}^{-1}\left(\sqrt{\frac{2}{\pi}} \cdot \frac{1}{\omega^2+4}\right) = 2e^{-2|x|}$$

7. $y''' + y' = t,\ t > 0\ ;\ y(0) = 1,\ y'(0) = 0,\ y''(0) = 0$

$$Y(s) = \mathcal{L}(y)(s) = \int_0^\infty e^{-ts} y(t)\, dt$$

$$s^3 Y(s) - s^2 y(0) + sY(s) - Y(0) = \frac{1}{s^2}$$

$$(s + s^3)Y(s) - (s^2 + 1) = \frac{1}{s^2}$$

$$Y(s) = \frac{\left(\frac{1}{s^2} + (s^2+1)\right)}{s + s^3}$$

$$= \frac{1}{s^3(1+s^2)} + \frac{1}{s}$$

Inverterar: $\mathcal{L}^{-1}\left(\frac{1}{s}\right) = t$

$$\mathcal{L}^{-1}\left(\frac{1}{s^3(1+s^2)}\right) = (\text{faltningsatsen})$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s^3}\right) * \mathcal{L}^{-1}\left(\frac{1}{1+s^2}\right)$$

$$= 2t^2 * \sin t.$$

$$= 2\int_0^t (t-x)^2 \sin x \, dx$$

8. $y'' - 6y' + 10y = \varphi(t) + te^t \; ; y(0) = 1, y'(0) = -1$

$\varphi(t) = 1, \; 1 < t < 2, \; \varphi(t) = 0$ annars.

Laplace:

$$s^2 Y(s) - s + 1 - 6sY(s) + 6 + 10Y(s)$$
$$= \mathcal{L}\varphi(s) + \frac{1}{(s-1)^2}$$

$$Y(s)(s^2 - 6s + 10) = \frac{1}{(s-1)^2} + 7 + s + \mathcal{L}\varphi(s)$$

$$Y(s) = \frac{1}{(s-1)^2(s^2-6s+10)} + \frac{7}{s^2-6s+10} + \frac{s}{s^2-6s+10} + \mathcal{L}\varphi(s)$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{(s-1)^2((s-3)^2+1^2)} + \frac{7}{(s-3)^2+1^2} + \frac{s}{(s-3)^2+1^2} + \frac{\mathcal{L}(\varphi(s))}{(s-3)^2+1^2}\right)$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s-3)^2+1^2}\right) = e^{3t}\sin t$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right) = t e^{-t}$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\cdot\frac{1}{(s-3)^2+1^2}\right) = \left(te^{-t}\right) * \left(e^{3t}\sin t\right)$$

$$\mathcal{L}^{-1}\left(\frac{s+7}{(s-3)^2+1^2}\right) = \mathcal{L}^{-1}\left(\frac{s-3}{(s-3)^2+1^2}\right) +$$

$$+\ \mathcal{L}^{-1}\left(\frac{10}{(s-3)^2+1^2}\right) = e^{3t}\cos t + 10\, e^{3t}\sin t$$

$$\mathcal{L}^{-1}\left(\mathcal{L}(\varphi(s))\cdot\frac{1}{(s-3)^2+1^2}\right) \overset{\text{Faltung}}{=}$$

$$\varphi(t) * \left(e^{3t}\sin t\right)$$