Lösningsförslag Naturvetenskaplig matematik A1, MMGK11 120830

1) a) $\int_0^1 x^3(1+x)dx = \int_0^1 x^3+x^4 dx = \left[\frac{x^4}{4}+\frac{x^5}{5}\right]_0^1 = \frac14+\frac15-(0+0) = \frac{9}{20}$ b) $\int 3\sin^2 x\cos x\,dx =$
$= \left[\begin{matrix} t=\sin x \\ \frac{dt}{dx}=\cos x \end{matrix}\right] = \int 3t^2 dt = t^3+C = \sin^3 x + C$ c) $\int \cos^3 x\,dx = \int \cos^2 x \cos x\,dx =$
$= \int (1-\sin^2 x)\cos x\,dx = \left[\begin{matrix} t=\sin x \\ dt = \cos x\,dx \end{matrix}\right] = \int 1-t^2 dt = t - \frac{t^3}{3}+C = \sin x - \frac{\sin^3 x}{3}+C$

2) $y' + 2xy = 2x$, $y(0)=2$ IF: $e^{\int 2x\,dx} = e^{x^2} \Rightarrow \frac{d}{dx}(e^{x^2}y) = 2xe^{x^2} \Rightarrow$
$e^{x^2}y = \int \frac{d}{dx}(e^{x^2}y)dx = \int 2xe^{x^2}dx = \left[\begin{matrix} t=x^2 \\ dt = 2x\,dx \end{matrix}\right] = \int e^t dt = e^t + C = e^{x^2}+C \Rightarrow y = 1+Ce^{-x^2}$
$\Rightarrow y(0) = 1+Ce^0 = 1+C \Rightarrow C=1 \Rightarrow y(x) = 1+e^{-x^2}$

3) $y''-4y'+3y = e^{2x}$ $y(0)=y'(0)=0$ $y=y_p+y_h$ Kar. ekv $0=r^2-4r+3 \Rightarrow$
$r_{1,2}=1,3 \Rightarrow y_h = C_1e^x+C_2e^{3x}$ Ansätt $y_p = x^m Ce^{2x} = [m=0] = Ce^{2x}$
$y_p' = 2Ce^{2x}$, $y_p''=4Ce^{2x}$ Insättning $\Rightarrow e^{2x} = y_p''-4y_p'+3y_p = 4Ce^{2x}-4\cdot 2Ce^{2x}+3Ce^{2x} =$
$= e^{2x}(4C-8C+3C) \Rightarrow -C=1 \Rightarrow C=1 \Rightarrow y_p = e^{2x}$ ∴ $y = e^{2x}+C_1e^x+C_2e^{3x}$
$y' = 2e^{2x}+C_1e^x+3C_2e^{3x}$ och begynnelsevillkor $\Rightarrow 0 = 1+C_1+C_2$, $0=2+C_1+3C_2$
∴ $C_2 = -\frac12 = C_1 \Rightarrow y = e^{2x} - \frac12 e^x - \frac12 e^{3x}$

4) a) $\ell = \sqrt{(4-1)^2+(5-1)^2} = \sqrt{9+16} = \sqrt{25} = 5$ b) $y = kx+m$ $k = \frac{\Delta y}{\Delta x} = \frac{5-1}{4-1} =$
$= \frac43 \Rightarrow y = \frac43 x + m$ och $1 = y(1) = \frac43\cdot 1 + m \Rightarrow m = -\frac13$ ∴ $y = \frac43 x - \frac13$
$\ell = \int_1^4 \sqrt{1+(y'(x))^2}\,dx = \int_1^4\sqrt{1+\left(\frac43\right)^2}\,dx = \sqrt{\frac{9+16}{9}}\int_1^4 dx = \frac53(4-1) = 5$

5) $\cos 2v = \cos(v+v) = \cos^2 v - \sin^2 v = 1-2\sin^2 v \Rightarrow \cos x - 1 = -2\sin^2\frac{x}{2}$
∴ $L = \lim_{x\to0} \frac{\cos x - 1}{x\sin x} = \lim_{x\to0}\frac{-2\sin^2\frac x2}{x\sin(2\cdot\frac x2)} = \lim_{x\to0}\frac{-2\sin^2\frac x2}{x\cdot 2\sin\frac x2\cos\frac x2} = \lim_{x\to0}\frac{-\sin\frac x2}{x\cos\frac x2}\cdot\frac{\frac12}{\frac12} =$
$= \lim_{x\to 0}\frac{-\sin\frac x2}{\frac x2}\cdot\frac{1}{\cos\frac x2}\cdot\frac12 = -1\cdot1\cdot\frac12 = -\frac12$

6) $I = \int e^{ax}\sin bx\,dx = [PI] = \frac{e^{ax}}{a}\sin bx - \frac ba\int e^{ax}\cos bx\,dx = [PI] = \frac{e^{ax}\sin bx}{a}$
$- \frac ba\left(\frac{e^{ax}}{a}\cos bx + \frac ba\int e^{ax}\sin bx\,dx\right) = \frac{e^{ax}}{a^2}(a\sin bx - b\cos bx) - \frac{b^2}{a^2}I$ Bootstrapping $\Rightarrow$
$\left(1+\frac{b^2}{a^2}\right)I = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2} \Rightarrow I = \frac{1}{a^2+b^2}(a\sin bx - b\cos bx)e^{ax}+C$ $a,b \neq 0$ konst $= C$

7) a) Om $\int_0^x ty(t)dt$ ∃ så gäller $y'(x) = \frac{d}{dx}\left(1+\int_0^x ty(t)dt\right) = xy(x)$ [illegible] om
$y(x) = \text{kont} \Rightarrow y'(x)\ \exists \Rightarrow y(x)$ kont

b) Om y kont så ∃ $\int_0^x ty(t)dt$ och $y' = \frac{d}{dx}\left(1+\int_0^x ty(t)dt\right) = xy(x)$
$\Rightarrow y'-xy=0 \Rightarrow$ IF: $e^{\int -x\,dx} = e^{-x^2/2}$ ∴ $\frac{d}{dx}(e^{-x^2/2}y) = e^{-x^2/2}\cdot 0 = 0 \Rightarrow$
$e^{-x^2/2}y = \int\frac{d}{dx}(e^{-x^2/2}y)dx = \int 0\,dx = C \Rightarrow y = Ce^{x^2/2}$ Vidare gäller $y(0) = 1+\int_0^0 ty(t)dt =$
$= 1+0 = 1 \Rightarrow 1 = y(0) = Ce^0 = C \Rightarrow y = e^{x^2/2}$