# MMGK11, del 2, 180315



1)

a) $\int \frac{1}{\cos^2 x}\,dx = \tan x + C$  b) $\int \frac{1}{\sqrt{x^2+4}}\,dx = \ln|x+\sqrt{x^2+4}| + C$

c) $\int \frac{1}{\sqrt{4-x^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{1-(\frac{x}{2})^2}}\,dx = \left[t=\frac{x}{2}\right] = \frac{2}{\cancel{2}}\int \frac{1}{\sqrt{1-t^2}}\,\cancel{2}\,dt =$

$= \arcsin t + C = \arcsin\left(\frac{x}{2}\right) + C$

d) $\int \frac{x}{\sqrt{4-x^2}}\,dx = -\frac{1}{2}\int \frac{-2x}{\sqrt{4-x^2}}\,dx = -\sqrt{4-x^2} + C$

e) $\frac{1}{x+x^2} = \frac{1}{x(x+1)} = [PBU] = \frac{A}{x} + \frac{B}{x+1} = [HP] =$

$= \frac{1}{x} - \frac{1}{x+1} \Rightarrow \int \frac{1}{x+x^2}\,dx = \int \frac{1}{x} - \frac{1}{x+1}\,dx = \ln|x| - \ln|x+1| + C$

f) $\int_0^{\pi/2} x e^x\,dx = [PI] = \left[x e^x\right]_0^{\pi/2} - \int_0^{\pi/2} e^x\,dx = \frac{\pi}{2}e^{\pi/2} - 0 - \left[e^x\right]_0^{\pi/2} =$

$= \frac{\pi}{2}e^{\pi/2} - (e^{\pi/2} - 1) = \left(\frac{\pi}{2} - 1\right)e^{\pi/2} + 1$

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2) a) $y' = \frac{2}{x}y + x^2 \Leftrightarrow y' + \left(\frac{-2}{x}\right)y = x^2$. IF: $e^{\int -\frac{2}{x}dx} =$

$= e^{-2\ln|x|} = e^{\ln|x|^{-2}} = \frac{1}{|x|^2} = \frac{1}{|x^2|} = \frac{1}{x^2}$ $\therefore$

$\frac{d}{dx}\left(\frac{1}{x^2}y\right) = \frac{1}{x^2}x^2 = 1 \Rightarrow \frac{1}{x^2}y = \int \frac{d}{dx}(x^2 y)\,dx = \int 1\,dx = x + C$

$\therefore$ $y = x^2(x+C) = x^3 + Cx^2$

b) $y' = 2xy^2$ Vi ser att $y=0$ är eventuellt singulär lösning. Om $y \neq 0$ får vi $\frac{1}{y^2}\frac{dy}{\cancel{dx}} = 2x\,dx \Rightarrow$

$\Rightarrow \int \frac{1}{y^2}\,dy = \int 2x\,dx \Rightarrow -\frac{1}{y} = x^2 + C \Rightarrow y = -\frac{1}{x^2+C}$

$= \frac{1}{C-x^2}$, $C \in \mathbb{R}$ $\therefore$ $y = \begin{cases} \frac{1}{C-x^2}, C \in \mathbb{R}, & \text{allmän lösn.} \\ 0 & \text{singulär lösn.} \end{cases}$

c) $y'' - 3y' + 2y = 0$
Kar. ekv. $0 = r^2 - 3r + 2 =$

$= (r-1)(r-2) \Rightarrow y = y_h = C_1 e^x + C_2 e^{2x} \Rightarrow y' = C_1 e^x + 2C_2 e^{2x}$

$\therefore \begin{cases} C_1 + 2C_2 = y'(0) = 0 \\ C_1 + C_2 = y(0) = 0 \end{cases} \Rightarrow C_1 = 0 = C_2 \therefore y = 0e^x + 0e^{2x} = 0$

$\therefore y \equiv 0$

3) $y'' + y' - 2y = e^{\pm x}$, Linjär $\Rightarrow y = y_p + y_h$

$y_h$: kar. ekv.: $0 = r^2 + r - 2 = (r-1)(r+2) \Rightarrow y_h = C_1 e^x + C_2 e^{-2x}$

$y_p$: a) Ansätt $y_p = x^m C e^{-x} = (m=0) = Ce^{-x}$
$\Rightarrow$ (derivering och insättning i ekv) $\Rightarrow e^{-x} =$

$= y_p'' + y_p' - 2y_p = Ce^{-x} - Ce^{-x} - 2Ce^{-x} = -2Ce^{-x}$

$\Rightarrow -2C = 1 \Rightarrow C = -1/2 \Rightarrow y_p = -\frac{1}{2}e^{-x}$

$\therefore$ $\boxed{y = -\frac{1}{2}e^{-x} + C_1 e^x + C_2 e^{-2x}}$

b) Ansätt $y_p = x^m C e^x = (m=1) = Cxe^x \Rightarrow$
$\Rightarrow$ (derivering och insättning) $\Rightarrow e^x = y_p'' + y_p' - 2y_p =$

$= 2Ce^x + y_p + Ce^x + y_p - 2y_p = 3Ce^x \Rightarrow$

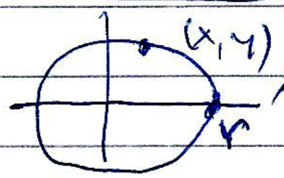
$3C = 1 \Leftrightarrow C = 1/3 \therefore y_p = \frac{x}{3}e^x \therefore$ $\boxed{y = \frac{x}{3}e^x + C_1 e^x + C_2 e^{-2x}}$

Lösn MMGK11, del 2, 180315

4) $\cos t = 1 - \frac{t^2}{2!} + \mathcal{O}(t^4) \Rightarrow$

$$\lim_{x\to 0} \frac{1-\cos(x^2)}{(1-\cos x)^2} = \lim_{x\to 0} \frac{\frac{x^4}{2!} + \mathcal{O}(x^8)}{\left(\frac{x^2}{2!} + \mathcal{O}(x^4)\right)^2} =$$

$$= \lim_{x\to 0} \frac{\cancel{x^4}\left(\frac{1}{2} + \mathcal{O}(x^4)\right)}{\cancel{(x^2)^2}\left(\frac{1}{2} + \mathcal{O}(x^2)\right)^2} = \frac{\frac{1}{2}+0}{\left(\frac{1}{2}+0\right)^2} = 2$$

5) 


$x^2+y^2=r^2 \Rightarrow V = 2\int_0^r \pi\left(\sqrt{r^2-x^2}\right)^2 dx =$

$$= 2\pi \int_0^r r^2 - x^2\,dx = 2\pi\left[r^2x - \frac{x^3}{3}\right]_0^r = 2\pi\left(r^3 - \frac{r^3}{3}\right) = \frac{4\pi r^3}{3}$$

6) $$\int_{-\pi/4}^{\pi/4} (x+1)|\tan x|\,dx = \begin{bmatrix} \tan x \text{ udda} \\ |\tan x| \text{ jämn} \\ x|\tan x| \text{ udda} \end{bmatrix} = \int_{-\pi/4}^{\pi/4} x|\tan x|\,dx +$$

$$+ \int_{-\pi/4}^{\pi/4} |\tan x|\,dx = 0 + 2\int_0^{\pi/4} \tan x\,dx = \begin{bmatrix} t = \cos x \\ \frac{dt}{\cancel{dx}} = -\sin x\,dx \end{bmatrix} =$$

$$= -2\int_1^{1/\sqrt{2}} \frac{1}{t}\,dt = 2\int_{1/\sqrt{2}}^1 \frac{1}{t}\,dt = 2\left[\ln|t|\right]_{1/\sqrt{2}}^1 =$$

$$= 2\left(\ln 1 - \ln\frac{1}{\sqrt{2}}\right) = 2\left(0 - (-\ln\sqrt{2})\right) =$$

$$= 2\cdot\frac{1}{2}\ln 2 = \underline{\underline{\ln 2}}$$

7) (*) $y'' - (y')^2 + y'(y-1) = 0$, $y(0) = y'(0) = 2$

I ekvationen saknas $x$ ⇒ ansätt $z = z(y) = y'$ ⇒

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(z) = \frac{dz}{dy}\frac{dy}{dx} = z\frac{dz}{dy}$$

∴ (*) ⇒ $z\frac{dz}{dy} - z^2 + z(y-1) = 0$ Vi kan anta att sökt lösning $y \not\equiv 0$ ty vi söker lösning med $y'(0) = 2 \neq 0$ så $z \neq 0$ ∴ $\frac{dz}{dy} - z = 1 - y$

som är 1:a ordn. linjär för $z$ med IF:

$$e^{\int -1\,dy} = e^{-y} \Rightarrow \frac{d}{dy}(e^{-y}z) = (1-y)e^{-y} \Rightarrow$$

$$e^{-y}z = \int \frac{d}{dy}(e^{-y}z)\,dy = \int (1-y)e^{-y}\,dy = [PI] =$$

$$= (1-y)(-e^{-y}) - \int (-1)(-e^{-y})\,dy = ye^{-y} - e^{-y} + e^{-y} + C = ye^{-y} + C \Rightarrow y' = z = y + Ce^{y}$$

och $y'(0) = y(0) = 2 \Rightarrow C = 0 \Rightarrow y' = y \Rightarrow$

⇒ (linjär och/eller separabel) ⇒ $y = C_1 e^x$, $C_1 \in \mathbb{R}$

∴ $2 = y(0) = C_1 e^0 \Rightarrow C_1 = 2 \Rightarrow$ sökt lösning är $\boxed{y = 2e^x}$ (koll: .....)

8) Se kurslitteraturen.