Analytic Geometry and Linear Algebra.

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Now the basic objects of geometry can be described in terms of coordinates.

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5. The *angle* between two directions v and w is given by

$$\arccos(\frac{v \cdot w}{|v||w|}),$$

where $v \cdot w = \sum v_j w_j$.

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There is an extra (unexpected ?) bonus with the translation to coordinates: We can do geometry in any dimension, and it is in principle as easy as in two dimensions. Here is an example of this:

The method of least squares

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Two numbers, a and b determine the line $y = ax + b$. Instead of trying to solve the *overdetermined* system of equations

$$y_j = ax_j + b$$

we try to minimize the error

$$\epsilon^2 = \sum_j (y_j - (ax_j + b))^2$$

over all choices of a and b.

Let $\mathbf{x} = (x_1, ...x_n)$, $\mathbf{1} = (1, 1...1)$ (**two points in $\mathbb{R}^n$!!**) and let P be the twodimensional plane

$$P = \{a\mathbf{x} + b\mathbf{1}; a, b \in \mathbb{R}\}.$$

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How do we find it?

It is clear from a figure that the minimum will occur in a point (a_0, b_0) such that $\mathbf{y} - (a_0\mathbf{x} + b_0\mathbf{1})$ is perpendicular to any vector in the plane. (Excercise: prove this!).

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$$[\mathbf{y} - (a_0\mathbf{x}+b_0\mathbf{1}] \cdot \mathbf{x} = 0, \quad [\mathbf{y} - (a_0\mathbf{x} + b_0\mathbf{1}) \cdot \mathbf{1}] = 0.$$

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This is a homogenous system of two equations and two unknowns which always has a solution. Observe that a_0 and b_0 are the unknowns, and $\mathbf{x}$, $\mathbf{y}$ are given!

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Notice that it solves a problem in the plane by using geometry in n dimensions, where n is the number of points and can be arbitrary big. The method of least squares was probably first used by Gauss, who applied it to find a 'lost planet'.

The astronomers had found a dwarf planet Ceres and recorded its positions for some time, but suddenly they could not see it anymore, because it was close to aligned with the sun. Gauss extrapolated the position from the known trajectory and could tell the astronomers where to look.

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A similar problem arises when we try to compress a picture with many pixels to few kilobytes. This is where ‘least sums’ have proved to be surprisingly useful.

One central topic in linear algebra is the solution of linear systems of equations

$$\begin{array}{c} a_{11}x_1 + ...a_{1n}x_n = y_1 \\ a_{21}x_1 + ...a_{2n}x_n = y_2... \\ a_{m1}x_1 + ...a_{mn}x_n = y_m \end{array},$$

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$$

or

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where A is the coefficient matrix of the system.
Here is the most important theorem in that context. We think of A as a linear map $x \to Ax$ from $\mathbb{R}^n$ to $\mathbb{R}^m$. Recall that $Ker(A) = \{x; Ax = 0\}$ and $Im(A) = \{Ax; x \in \mathbb{R}^n\}$; they are both linear subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

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The statement and the proof hinges on the notion of dimension. A linear space, like $\mathbb{R}^n$ has many different bases, but they have all the same number of elements. (Exercise: Prove this!) This is the dimension of the space. Say the dimension of $Ker(A)$ is k, and let $e_1, ...e_k$ be a basis. We can find vectors in $\mathbb{R}^n$, $f_1, ...f_{n-k}$ that complete $e_1, ...e_k$ to a basis of $\mathbb{R}^n$. Let F be the linear span of $f_1, ...f_{n-k}$. Then the restriction of A to F is injective (why?).

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The theorem can be reformulated in the following way. Let $G = \mathbb{R}^m / Im(A) := coker(A)$. Then

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The advantage with this formulation is that the kernel and the cokernel may have finite dimensions even if A acts on an infinite dimensional space. If $A : V \to V$ where V is a vector space of finite dimension, then the index is always zero. This is not always the case in infinite dimensions as we shall see later. The index is an important object to study in the theory of partial differential equations, when A is a differential operator.

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be a *quadratic form*. If $A = (a_{ij})$ we may write

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and we may assume that A is symmetric. If we change basis in $\mathbb{R}^n$, $x = My$, where M is an invertible matrix, we have

$$Q(x) = y^t M^t A M y = Q'(y).$$

We now have the second important theorem of linear algebra:

Theorem

We may find an (orthonormal) M such that

$$Q'(y) = \sum \lambda_j y_j^2.$$

This is the *Spectral Theorem*. If we interpret A as a linear operator, $A' = M^{-1}AM$ is the matrix for the same operator in the new basis, where y are coordinates. But, since M is orthonormal, $M^t = M^{-1}$. hence the theorem says that we change coordinates so that A' is the diagonal with eigenvalues λ_j.

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We are now ready to discuss the corresponding facts in infinite dimension.

Infinite dimension and Hilbert space.

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exists.
In other words, there is an element u in the space such that

$$\|u - \sum^{n} u_j\| \to 0.$$

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Example 2 is complete, Example 1 is not.

Theorem

Every Hilbert space V has an orthonormal basis, i e there is an orthonormal set of vectors $\{e_\alpha\}_{\alpha \in A}$ such that any vector in V can be written

$$x = \sum_A c_\alpha e_\alpha,$$

and

$$\|x\|^2 = \sum_A |c_\alpha|^2.$$

Theorem

Every Hilbert space V has an orthonormal basis, i e there is an orthonormal set of vectors $\{e_\alpha\}_{\alpha \in A}$ such that any vector in V can be written

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Briefly, there is only one Hilbert space.

Corresponding to matrices we now have linear maps, or *operators*, $A : V \to V$. Let $B = \{x; \|x\| \leq 1\}$ be the unit ball in V.

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2. We say that A is compact if $A(B)$ is compact.

3. There is also a weaker notion of closed linear map that I will not give.

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Let A be a compact operator on a Hilbert space H. Assume A is selfadjoint, i e

$$(Ax, y) = (x, Ay).$$

Then the quadratic form (Ax, x) can be diagonalized. This means that there is an orthonormal basis (e_j) of eigenvectors of A.

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Then the quadratic form (Ax, x) *can be diagonalized. This means that there is an orthonormal basis* (e_j) *of eigenvectors of A. Moreover, the eigenvalues* λ_j *tend to 0.*

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This corresponds to the linear map $Af = -f''$, which is not bounded and certainly not compact. Nevertheless the theorem applies, essentially because the inverse of A is compact. Hence there is a basis of eigenvectors, namely $e_j(\theta) = e^{ij\theta}$.

The eigenvalues of A are $\lambda_j = j^2$ and any function can be written

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This ‘explains’ Fourier analysis but has much wider scope. E g we can consider instead a domain D in the plane and the Hilbert space of functions that are square integrable on D, with the quadratic form

$$\int_D |\nabla f|^2.$$

Let us continue with the last example and consider the plane in our Hilbert space

$$P := \{f; f = g \ on \ \partial D\},$$

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Then f_0 must be 'orthogonal' with respect to $\mathcal{Q}$ to any vector u in the plane $P - f_0$. Such functions u are of the form $u = f - f_0$, where both f and f_0 equal g on the boundary of D, i e they are just functions that vanish on the boundary of D.

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Hence

$$0 = Q(f_0, u) = \langle(-\Delta f_0), u\rangle,$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the *Laplace operator*.

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If this holds for all u that vanish on the boundary, $\Delta f_0 = 0$ (and $f = g$ on the boundary). So, we have solved *Dirichlet's problem*.

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$$(\lambda I - T)f = g,$$

where λ is a number and T is a compact operator. A typical compact operator is

$$Tf(x) = \int K(x, y)f(y)dy,$$

where K is continuous. This is the integral version of an operator given by matrix multiplication.

Theorem

*(**The Fredholm alternative**) Let*

$$Tf(x) = \int K(x,y)f(y)dy,$$

where K is continuous. Then, for any complex number λ, either the equation

$$(\lambda I - T)f = g$$

has a solution f for any choice of g, or the equation

$$(\lambda I - T)f = 0$$

has a non trivial solution.

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Fredholms article was published in 1903 and inspired Hilbert's general theory on integral equations and the solvability of 'equations in infinitely many variables'. (1912).

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The next big step was John von Neumann's general theory of Hilbert spaces (he introduced that name) as a foundation of quantum mechanics in 1932 (when von Neumann was 29 years old).

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In this way we can see Hilbert space as the mathematical theory of quantum mechanics, similarily to how Riemannian geometry is the mathematics of the theory general relativity. We shall next turn to the mathematics of classical mechanics, i e calculus.