Calculus.

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The calculus of integrals and derivatives is often presented as having started with Newton and Leibniz, but of course there were many predecessors. We have already seen how versions of of integration was already used in antiquity, by Euxudus and Archimedes. Another early theorem that today would be interpreted as a theorem about integrals is *Cavalieri's principle* (1635).

It says that if we have two regions in the plane, both defined by the graph of functions

$$R_i = \{(x, y); g_i(x) < y < f_i(x)\},$$

and if

$$f_1(x) - g_1(x) = f_2(x) - g_2(x)$$

for all x then the areas of the two regions are equal.

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If a is a local maximum there will be two distinct points x_1 and x_2 close to a such that $p(x_1) = p(x_2)$ (one on each side of a).

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Since this holds for points arbitrarily close to a it must hold for $x_1 = x_2 = a$. We get

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Apparently this was first noted my Newton's teacher, Barrow. (Barrow was reputedly a 'wild character', sent off to academic studies by his wealthy father who did not want him involved in the family business. As subject of study he choose – theology. Theology lead to chronology and attempts to reconcile the age of the earth according to the bible with known historical records. Chronology in turn lead to astronomy and, then, mathematics.)

Barrow's results were however not as clearly formulated as in the succint equation above. The honor of having discovered the fundamental theorem of calculus is instead ascribed to Newton and Leibniz. The story is complicated by the fact that Newton did not publish his work on derivatives until fairly late, in 1693. By that time, Leibniz had already published his version of the theory, in 1684, which lead to a long controversy between the two.

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Newton is said to have stated that any person in science must make a choice: Either to publish nothing, or to devote all his life to a struggle for priority. According to the russian mathematician Arnold, a great admirer of Newton's, Newton made the worst of these alternatives; he published almost nothing – *and* was constantly struggling for priority.

Most of Newton's most well known work was carried out between 1665 -1667, during the plague years. (He was born in 1642). This includes, probably, his work on the method of derivatives and also the Newtonian theory of classical mechanics that was not published until 1687, in his Principia Mathematica.

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The story of how the Principia of Newton came to be has many interesting parts. In 1679 Newton was approached by Hooke, who asked Newton if he could give a mathematical proof that the inverse square law of gravitation forces the planets to move in elliptic trajectories.

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In principia Newton formulated what is now known as Newton's laws, essentially the law of acceleration

$$\vec{F} = m\vec{a}$$

and the law of gravitation

$$F = \frac{mM}{r^2}$$

or rather

$$\vec{F} = -mM\frac{\vec{r}}{r^3},$$

(the inverse square law).

He then went on to draw all sorts of consequences using mathematical analysis, including the elliptic shape of planetary orbits.

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Newton substituted the formula of an ellipse for r and saw that it fit. This method of solution is basically ok if you know uniqueness – which was probably also obvious to Newton.

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Let $x(t)$ where t runs from a to b be a curve, such that it is the shortest curve between $A := x(a)$ and $B = x(b)$. We may assume that x is parameterized by arc length. Let

$$L(s) = \int_a^b |\dot{x}(t) + s\dot{y}(t)|dt,$$

where $y(a) = y(b) = 0$.

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where $y(a) = y(b) = 0$. Then $L'(0) = 0$.

But

$$L'(0) = \int_a^b \frac{\dot{x} \cdot \dot{y}}{|\dot{x}|} dt = \int_a^b \dot{x} \cdot \dot{y} \, dt,$$

since $|\dot{x}| = 1$ when the curve is parametrized by arc length.

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Similarily one can show that a circle is the curve of a given length that encompasses the greates area. (Much more difficult though.) But all these methods presuppose that we have a curve that gives the minimum. Such problems were not solved until much later, after the rigorous introduction of the real number system, limits and the supremum axiom, by Cauchy, Weierstrass and Dedekind.

One of the most important applications of the calculus of variations is the *principle of least action* in mechanics.

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$$L(x, \dot{x}) := \frac{m\dot{x}^2}{2} - V(x),$$

where V is the *potential energy*. Thus, the action is the *difference* between the kinetic energy and the potential energy, as opposed to the total energy which is the *sum* of kinetic and potential energy.

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Newton's laws can be written

$$m\ddot{x}(t) = \vec{F} = -\frac{\partial V}{\partial x}.$$

This can be written elegantly in terms of the action:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}.$$

So far this is just a rewrite. Now introduce the total action of a curve $\gamma = x(t), a < t < b$:

$$S(\gamma) = \int_a^b L(x(t), \dot{x}(t))dt.$$

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$$0 = (d/ds)|_{s=0} S(\gamma + sy(t)) = \int_a^b y\frac{\partial L}{\partial x} + \dot{y}\frac{\partial L}{\partial \dot{x}} dt,$$

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After an integration by parts in the second term this means precisely that

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i e Newton's equations!

So, a curve $x(t)$ satisfies Newton's equations when it minimizes the action integral (or, more precisely, is stationary for the action integral). This is called the principle of least action.

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This was a few years after the births of Fourier and Gauss, whose work would mark a new era in mathematics.