Euclidean Geometry

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We will focus on the postulates which with modern eyes should not be considered as evident, but rather as assumptions:

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(4) All right angles are equal to each other.

(5) Through any point outside a given line there is exactly one line which does not intersect the given line. (PA)

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Hence

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Erathostenes (276-195 BC) computed the circumference of the earth. When Columbus set out to sail to India it had become a matter of debate whether the earth was round.

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Non euclidean Geometry

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Divide the (x, y)-plane into the upper half plane ($y > 0$) and the lower half plane ($y < 0$). Let us say that the distance between two points in the upper half plane is the usual distance, whereas the distance between two points in the lower half plane is twice the usual one.

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This is of course the standard model for the refraction of light between a denser and a lighter medium. (Exercise: Show that $\sin \alpha / \sin \beta = a$ if the distances in the lower half plane are a times the distances in the upper half plane.)

We can generalize this model. Recall that if $\gamma(t), t \in [a, b]$ is a (piecewise) smooth curve its length is

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preserves length. This gives (using some basic facts about Möbius transformations) that the geodesics are the half circles (or line segments) that intersect $|z| = 1$ in two right angles:

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The line segment $[-1, 1]$ is a geodesic. It is a straight line that intersects the unit circle at right angles. Therefore, by a result from complex analysis, its image under M is either a line or a circle, intersecting the image of the unit circle at right angles. But, the image of the unit circle is the unit circle.

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To see this, take a half circle C, not containing the origin, intersecting the unit circle at right angles. This is our 'straight line'. Now look at the origin. This is our point outside the 'straight line'. Clearly there are many diameters of the circle that do not intersect C. Thus, PA does not hold, but one can check that the other postulates do hold.

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To see this, take two half lines through the origin. This gives us two sides of a triangle. Let the third side be a half circle intersecting the unit circle at right angles outside the sector formed by the two half lines. Inspection of the figure shows that the sum of angles is smaller than π. Note however that if the half circle is close to a diameter, so that our triangle is very small, then the sum of angles is *almost* π. So, on small scales, the new geometry is almost Euclidean.

More generally still we can define lengths by

$$L_g(\gamma) = \int_a^b \sqrt{\sum g_{ij}(x(t))\dot{x}_i(t)\dot{x}_j(t)}dt,$$

where $g(x) = (g_{ij}(x)$ is a matrix valued function that is positive definite at any point.

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For instance, if we look at the surface

$$\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 = 1\},$$

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If (g_{ij}) is the metric, we first define the Christoffel symbols

$$\Gamma_{ij}^{m} = 1/2 \sum_{k} g^{mk} \left(\frac{\partial g_{ki}}{\partial x_j} + \frac{\partial g_{kj}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k} \right).$$

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Then we simply have

$$R_{ijk}^{l} = \frac{\partial \Gamma_{ik}^{l}}{\partial x_j} - \frac{\partial \Gamma_{ij}^{l}}{\partial x_k} + \sum_s \Gamma_{js}^{l} \Gamma_{ik}^{s} - \Gamma_{ks}^{l} \Gamma_{ij}^{s}.$$