First Lecture

Thales and the Beginnings

When to begin with mathematics? Its origin is lost in prehistory and if you are sympathetic enough you can even attribute to animals mathematical skills, although hardly of a conscious kind. And nature, of course, abounds in mathematics. So let us begin in Media Res.

One name to commit to memory is Thales. He was a Greek philosopher, mathematician and merchant, although not necessarily in that order. People of the past were never unduly specialized. He lived in the 7th century before Christ, thus a few centuries ahead of Plato and other figures of the Greek Civilization at its ripest, and hence at the verge of its demise.

The Greeks are the heroes of our Western Civilization and for many centuries the study of the Classics carried with it high prestige and to be educated meant that you could read Classical Greek. Thus the subject may have been studied to its death, after all there are only so many documents to read and ponder, and most educated people of today probably know less about Classical Greek than reluctant school-boys did a few generations ago. Educational projects imposed from above invariably invite resentment, but for us blessed with ignorance, an encounter with the Greek may be very refreshing.

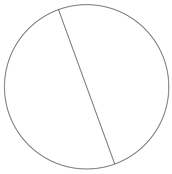
So here we have that race of Ancient men sprung upon the scene, God knows from where. Naturally any talk about race today is sensitive, and more to the point speculative and irrelevant, however, there is no denying of a common language as a unifying and defining feature of a population, so deeply implanted that it survived a temporary collapse of Civilization[1]. The Greeks were traders. They established colonies all over the Mediterranean and the Black Sea. They acquired wealth and leisure and an independence of spirit making them rather immune to religious dogma, which tends rarely to develop in poly-theistic religions[2]. All kinds of arguments can be found to explain the sudden and unprecedented rise of science in the Greek culture. So let us return to Thales.

I noted he was a businessman and apparently of some astuteness, if we are to believe anecdotes which are told about him. Sensing that the harvest of olives would be particularly good he laid his hands on all the olive presses he could get hold on, so that he during harvest time could rent them out at exorbitant prices. This is the kind of smartness that people in general can relate to. That according to another anecdote he fell into a well, while walking staring at the stars, is more apt to provoke derision among the same than admiration and desire for emulation. He is thought to be the first mathematician in the sense that he introduced deductive thinking into mathematics. Even that is by hearsay, we know what Thales reasoned about, but we do not know of any explicit examples, which makes it hard to judge him. The point is that although the script has a long tradition, actually writing things down was something else.

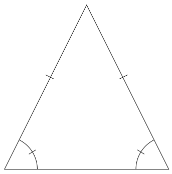
Few things were written down in the old Greek times, especially as far back as Thales. The necessary material for it was scarce. Furthermore much of the written documentation from those times have been lost over the ages. Some of it preserved only through translation. Thus the Greek legacy is a very fragmented one, tantalizing pieces of which have intrigued generations of later scholars. We know enough to be able to appreciate what has been lost. Yet, Greek society was an oral one. You did not learn by reading books, you learned by word of mouth, necessitating extensive travels around the known world. Thales had traveled widely getting into contact with traditional mathematics, especially the Egyptian kind.

'Geometry' is a Greek world based on the Egyptian practice of measuring land. Some European languages such as Dutch have made translated loans of the term (Meetkunde) but we mostly have adopted the Greek word wholesale. The Egyptian practice was purely practical, taxes were levied proportionally to the amount of land. Those holdings were periodically affected by floodings of the Nile and it was of some importance to be able to measure the changes. The Egyptians used a medley of methods of computations, seldom accurate, but accuracy was of less importance than getting a figure for taxation purposes. Thus such ambitions that would go beyond, would easily have been dismissed as 'academic'. With Thales the interest in geometry took on a more fundamental aspect and it was done, so to speak for its own sake. This is very important, any inquiry that is merely practical, will never develop beyond its preset limits, because the questions you can ask are bound to be limited and superficial. There needs to be a curiosity which probes deeper. Such a curiosity the Greeks possessed. This is why we admire and idealize them.

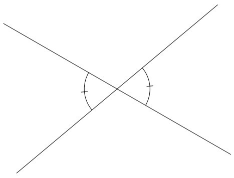
Here we give some examples of what it is reported that Thales had done and considered.



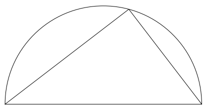
Every diameter of a circle cuts it in two equal parts.



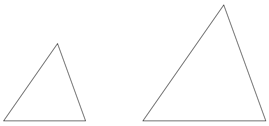
The angles at the base of an isosceles triangle are equal.



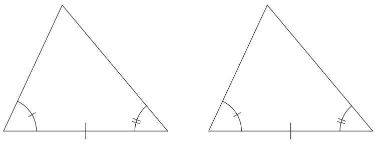
When two lines intersect the vertical angles are equal.



The angle in a semi-circle is a right angle.



The sides of similar triangles are proportional.



Two triangles are congruent if they have two angles and a side respectively equal.

What to make of them? As noted Thales was supposedly the first to reason deductively in mathematics and thus making it mathematics in the sense we now have of the word, but this is, as we have already noted, hearsay, we have no account of how Thales actually reasoned, we can only speculate.

Now one has to make one thing clear. Deductive reasoning is no guarantee for correctness. First, the reasoning itself maybe incorrect, and more seriously deductive reasoning cannot be done in a void, it has to be based on something, be it intuition or empirical findings. In fact the correctness of the reasoning is based on our rational intuition. The point of deductive reasoning is to provide arguments to make the process more transparent. By doing so it exposes itself to criticism. The point of criticism is not so much to reject as to improve. Providing arguments one may establish a dialog. Certain things may be agreed on, others may be matter for dispute. Any such discourse invites alternative arguments, new questions, maybe even tests and experiments. This is the essence of the scientific attitude. Thales had the right attitude. He encouraged his students not to take what he said as dogma, but to criticize it. According to Popper, with Thales started not only the mathematical but also the scientific tradition[3]. His students did not transmit so much his teachings as his attitude. This attitude survives until this day. If the attitude dies out it may never revive again. Mark that there was a time when the light of this attitude tended to flicker thus threatening to fade. It was kept alive through what is known as the Dark Ages by being documented, and then during the Renaissance it served as an inspiration. This historical presentation is of course simplified. Every historical presentation by necessity is, but it is accurate enough to illustrate the main point. Traditions need continuity. After a rupture there is no guarantee that they will occur spontaneously again, just as a species that has gone extinct may not be revived again.

It may be no coincidence that the scientific attitude started with mathematics. Unlike empirical inquiries it does not require an elaborate apparatus. It is an inquiry of thought verbally transmittable and aided by a crude sketching in sand or chalk onto stone, or whatever the technology at the time may provide. It is worth to point out that to this day the most effective means of transmitting mathematical knowledge and know-how (known as understanding) is by personal conversation. Mathematics also has both a deeper and a more tangible connection to the real world, than do other human cerebral exercises such as the law.

So after this digression we may return to Thales achievements. As mathematical achievements they strike us as rather pedestrian. Some of them appear rather obvious, such as the first one, and it is not clear how Thales argued for its validity, or why he found the reason to do so. One may speculate that he showed that any chord not through the center cut it in two unequal parts. The third example can be given an easy formal argument[4]. and the exercise seems to consist in how you could use a formal argument to support your intuition. The case by the right-angle in the semi-circle is more interesting. It is not entirely obvious and may strike most of us on first acquaintance as a surprise. Using the second statement it is easy to see that it is equivalent to the triangle in question having angular sum π, and thus that every triangle has such a sum. There is of course an easy argument for this, to the effect that a line hits parallel lines in equal angles (and the assumed possibility that through any point a line parallel to a given can be drawn) which Thales might have thought to be obvious. Most interesting though is the statement that similar triangles have proportional sides. This has striking practical consequences and lies at the heart of all the real-life applications of Euclidean Geometry. All our estimates of astronomical distances are based on it[5]. Did Thales have a proof of it, or was it merely a guess based on actually measuring triangles, noting that at the time of day when your own shadow is equal in length to that of your height, this is true for anything[6]. According to legend, this is how he measured the height of a pyramid in Egypt.

Pythagoras and Number Mysticism

Pythagoras was supposedly a student of Thales. Like with Thales no written documents of his have survived (maybe there never were any?). Thus his teachings and doctrines have to be fitted out of fragmentary evidence, not an enviable situation. It seems clear though that like Thales and others Pythagoras traveled extensively this being, as noted, the only way of getting an education. He finally settled in Southern Italy. The exact location being of secondary interest, and only mentioned to remind you that Greek Civilization was not confined to the Ægean Sea.

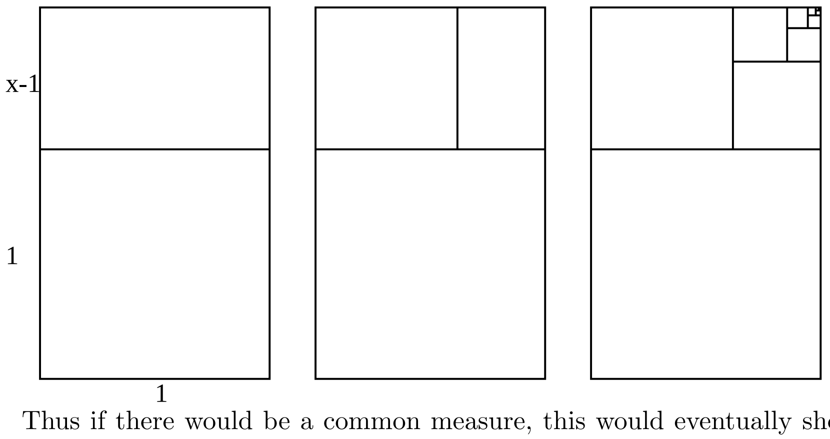
Pythagoras is mainly remembered for the number mysticism of his school, or rather his brotherhood, and Pythagoras Theorem. As to the latter Pythagoras may have had very little to do with the eponymous theorem. It was certainly known before his time and there is no indication that he provided a proof of it, but he certainly applied it. On the other hand you do not need to know Pythagoras theorem to get that the length of the diagonal is the square root of two, only that areas scale by the square, and that the square formed by a diagonal has twice the area of the original square. This leads to an embarrassing problem. The square root of two cannot be rational. The proof of this fact is very easy, and you have all seen it, and once you have seen it, you cannot forget it. Nevertheless let us try and recall it. You cannot solve the equation $p^2 - 2q^2 = 0$ in integers, which would be the case if $(p/q)^2 = 2$ would have a solution. You can also assume that not both p, q are even, because if so we can cut it out. But obviously p has to be even, because an odd square is always odd (note you do not need unique factorization for this you just check that $(2n+1)^2 = 2(2n^2+2n)+1)$[7] but then writing $p = 2p_0$ we get $2(2p_0 - q^2) = 0$ hence q is even too.

> By using unique factorization you can easily see that for an integer N $p^2 - Nq^2 = 0$ has a solution iff N is a square, because each prime factor has to have even multiplicity. If you do not know this fact, you have to treat laboriously each single case, as was reported in one of the dialogs

> of Plato, where the case of up to $N = 17$ was treated. Take the case of 17 you have in fact to check that 17 does not divide any of the sixteen numbers $1^2, 2^2, 3^2 \ldots 16^2$ which becomes a bit laborious, and certainly earns you credit for hard work and perseverance, and qualifies you for further grants to attack N=19 .

This had consequences and led to the notion of incommensurability. Given two lengths it is not unreasonable to suggest that there will be a common unit of length of which each is a multiple, in fact that they are commensurable, literally meaning to be jointly measured. This was a kind of scandal and it is rumored that the Pythagorean decided that geometry is more general than number theory.

We may pause here and make a short digression. The proof of the irrationality of $\sqrt{2}$ (and similar numbers) may be short, elegant and compelling. And as the British philosopher and historian R.G.Collingwood notes that any deductive reasoning is marked by this element of compulsion[8], but does it explain, or do you feel that it is a trick of some sort, the conclusions of which you are forced, or compelled, to accept? As we will discuss later in the course, you do not understand mathematics through chains of arguments, especially not formal ones, the meaning of mathematical statements only become clear when viewed in appropriate contexts, and especially seeing how they connect with other mathematical statements. There is another more geometrical proof of incommensarubility which is far more visual. It does not in its simplest form involve $\sqrt{2}$ but the solutions to the equation $x^2 - x - 1 = 0$, the so called Golden Ratio. Let us consider a triangle of width 1 and length x, we say that it is Golden if the remaining rectangle you get if you remove the square of side 1 is still Golden[9]. If so the procedure can be repeated indefinitely never coming to an end. This means exactly that the lengths 1 and x are incommensurable. If there would be a common measure between 1 and x, there would also be the same for x and $1 - x$



Thus if there would be a common measure, this would eventually show up after a finite number of steps.

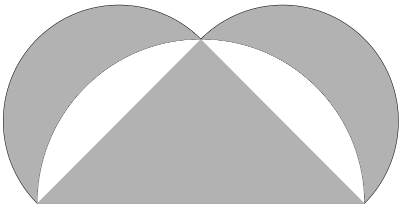
Now this proof is rather different from the first one. The first is very short, simple and elegant, but maybe it leaves you in the lurch. You feel almost tricked having no choice but to believe the conclusion, as we have already noted. The second proof on the other hand gives you a different kind of understanding, more visual and tangible, in a sense you pass through all those infinite number of steps, each one identical to the previous, which allows you the repetition. You may argue that this proof is not very rigorous nor convincing but is based on figures that very well may be misleading. This is a good objection, and shows that you are a Platonist in a very dutiful sense. Our senses can indeed play tricks on us. However, the proof may be by figure, but this is not exactly true, if I present it to youm and do not draw all those infinite number of rectangles, but you get the idea that it can in principle be done. Thus the crucial part of the proof is indeed cerebral, the idea of indefinite repetition, because the remaining rectangle has exactly the same shape as the initial, and hence we can repeat the argument. It is possible to make it all formal and thus to satisfy more exacting criteria, but admittedly this formalization is not obvious, but without this visual idea those pictures indicate you would not be able to do it. This points at a crucial fact of mathematical understanding. Deductive chains of reasoning tends only to give so called local understanding. One step compelling another to a conclusion you have no choice but to accept. It like being made to walk a route blindfolded, each step leading to another, and when reaching you destination, there is no reason to doubt its fact, although you really have no idea how to get there. To be honest many of the proofs you encounter in mathematics are like that. Thus it is nice once in a while to find a more conceptual proof, even if it is not as rigorous. Proofs like that were known to the Greeks. The one I showed you is the simplest one[10]. The problem is of course to show that there is such a measure, the case of $\sqrt{2}$ was easily exhibited as a diagonal in a square, thus it needs to be constructed, or equivalently to appear in a natural way, and indeed it will show up using diagonals in a regular pentagon. The first proof has additional advantages, it can be vastly generalized, while the second one requires ingenuity from case to case. Thus it is in a sense more 'artistic' with a greater emphasis of the individuality of each case.

The Pythagorean obsession with numbers had both a silly and a profound aspect. The silly aspect is to associate with various low numbers different properties, and maybe also the obsession with the shapes of different numbers such as triangular, squares and pentagons and notions of perfect numbers and amiable pairs, which smack too much of recreational mathematics and turned out to be ultimately sterile. The profound part, that everything can be explained by numbers, still very much dominates computer science and artificial intelligence. The idea that everything can be codified by numbers (such as the contents of your brain including its thoughts) and thus be copied and subjected to calculations, which in their turn can be codified by even bigger numbers. Of course given the primitive technology and knowledge of the Pythagoreans, examples of the explaining role of numbers are not so many among them, except by one notable example, which we will briefly explain, and which has earned Pythagoras the accolade of being the first mathematical physicist.

Some people are blessed (or cursed) by so called absolute pitch. This means they can identify a single note by its frequency. Whether it is congenital or can be acquired through patient training is not known, and its relation to 'musicality' is not clear. For most people they can only relate relative frequencies. If that ratio is 2 the sounds will in a sense sound 'the same'. We are thus talking about a cyclic group $\mathbb{R}^+/\{2^n\}$ in modern notation. Now to measure frequencies of sounds was beyond the capabilities of the Pythagoreans, on the other hand they could measure directly the lengths of strings giving rise to notes. If there was a simple relation such as $3:2$ (meaning that the numerators and denominators of the corresponding fractions were small) the two notes harmonized. This was a remarkable discovery directly relating an abstract numerical relation with a sensual impression. Now the difference between two notes of ratio 2:1 is called a scale. Traditionally in Western music this is discretized into eight intervals hanec referred to as an octave. Ideally they should be invariant under translation, meaning that the subsequent ratios are equal. This leads to an impossible mathematical problem having to do with the eight-root of two being irrational. Various compromises have been worked out. The old Pythagorean scale was gradually replaced by a so call well-tempered one in the 17th century or so, intended to make the spacing more uniform, at the expense of small denominators. There are many elementary connections with music and mathemaatics related to Pythagoras, but they are at a rather elementary level.

Hipparchus and the squaring of the circle and other Greek problems

The Greeks considered three problems of construction. One was the trisection of the angle, the bisection of one is easily effected by ruler and compass. Another one was the duplication of the cube, and the final one was the squaring of the circle. The first two were eventually solved by considering more general constructions, in particular by invoking other curves than straight lines and circles. The impossibility of construction by ruler and compass was not effected until the beginning of the 19th century, when it was shown in an elementary way that a cubic field cannot be generated by successive quadratic extension [11]. The impossibility of the third had to wait for the end of the 19th century when Lindemann showed that π is transcendental, i.e. not the root of any algebraic equation, in particular not one given by extension of quadratics. However certain figures bordered by curves, in fact by circular arcs could be squared. Using the fact that the areas of circles are proportional to the squares of their diameters it enabled Hipparchus of Chios to note that the shaded lunes (see figure below) had a total area equal to the shaded triangle, and from which it is not hard to find a square with the same area.

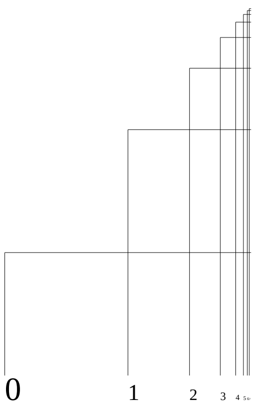


In fact the sum of the areas of the two smaller half-circles are equal to the area of the big half-circle, as areas of circles are proportional to the square of the diameters. Now from those two areas we subtract the common parts (those unshaded) and hence that remains have to have the same areas.

The remarkable thing of the example is that some rather complicated regions - the lunes, could have their areas so simply expressed. Would there not perhaps be some clever way of also finding a simple area for the circle? Hipparchus example was something of a fluke and as such a dead-end.

Zeno and his paradoxes

Zeno presented three paradoxes. They had to do about indefinite indivisibility and the problem of continuity. They can be paraphrased as follows:



Let the turtle and swift-footed Achilles have a race, with the turtle given a head-start. Will Achilles ever catch up with the turtle? Common sense tells us that this will always be the case, and we can easily cook up a formula in terms of the length of the head-start, and the constant velocities of the turtle and Achilles respectively to compute where and when this overtaking will take place. Zeno argued that Achilles could not in fact catch up by postulating an infinite number of positions P_n defined inductively as P_0 being Achilles initial position and P_{n+1} being the position of the turtle when Achilles is at P_n. To assume that Achilles catches up with the turtle means that in finite time Achilles passes through an infinite number of points, or if you want experiences an infinite number of events.

If the experience of each event would take a minimum amount of time, strictly positive, then indeed Achilles would not catch up, he would at each position P_n make a short stop and reflect something to the effect now I have reached the turtles position that was yonder a moment ago.

This is clearly a so called thought-experiment. In real physical life we cannot keep on determining the points with the increasing precision which is required. However, if we think in terms of distance quanta, i.e. that the continuous line is made up of small indivisible points adjacent to each other, new problems will occur. There will be time quanta and the velocities of the opponents would be measured by those units, and the actual positions they would occupy would not cover all the possible.

In modern mathematics we see little trouble here. It is a question of adding infinite geometric series which may converge even if an infinite number of terms are involved. As distances go to zero (i.e. distances between the two runners) so will the times needed to cover them. So what Zeno seems to say from a mathematical point of view is that before a certain time has elapsed, Achilles has not yet caught up. Big deal. Mathematicians hence tend to dismiss Zeno as if not mathematically irrelevant at least overcome. There is no paradox here to ponder. Philosophers tend to think otherwise. Maybe because they are mathematically unsophisticated?

A similar paradox has to do with movement itself. Can something, like an arrow, move in fact? Before it reaches position P_0 it has to reach a position P_1 halfway to P_0 and so on. This is of course a species of pedantry with which we are familiar in many other contexts, and known as infinite regress, when it comes to logical reasoning. What is meant by continuous movement? Can you speak about movement at a time instance at all? One naive thing is to think in terms of a sequence of still frames, in each the arrow is at rest, yet at each subsequent one it has moved. Twenty-four frames a second seems to be enough for the human brain to experience the illusion of continuous movement. This remark has relevance to physiology not mathematics but is nevertheless not without interest. Imagine that we could have needed hundreds of frames. Movies would have been much more involved and the development of the movie-industry would have been retarded. Also movies would have demanded much more storage space in the computer. The number 24 seems to be peculiar to humans, flies apparently would detect a flickering light where we would only notice a continuous one. The interpretation seems to be that they are living at a quicker pace, thinking quicker, maybe because their thoughts are simpler. But let us not digress.

Plato and Mathematical Platonism

Plato was no mathematician of renown. No result of any importance is attributed to him. On the other hand his influence on mathematics was very important, not just at his time, but also for posterity. He, if anyone, deserves to be acclaimed as the patron saint of mathematics. Famously there was a sign at the entrance to the Academy, that no one unversed in geometry should be allowed to enter. The idea of using mathematics as a filter has a long tradition and it was revived at the time of Napoleon, when the various State Ecoles were founded, ostensibly to supply the Government with competent administrators and engineers, especially military such. Mathematics was accorded high status in the educational systems in the West at least until the end of the 20th century, and still so very much in modern France. Plato thought that mathematics was a very good training for the mind to deal with the underlying forms. In fact the proper objects of mathematics were not to be found in the world of sensory experience, the lines that one could draw in the sand, or chalk on a board, were but poor representations of the real thing, the underlying mathematical object. While Plato's idealism is considered as outmoded in current philosophical circles, and the Platonist is considered with amused condension (at best), mathematics is the last stronghold of Platonism. In fact nothing that humans have encountered seem more fit to illustrate Plato's ideas. Mathematicians tend to be Platonists, at least in an unsophisticated non-philosophical mood, convinced that the objects they manipulate are real, if impossible to physically pin down, and that they are engaged in exploration and discovery not in mere invention and convention. This not to say that there is no invention or convention in mathematics, only that inventions have unintended consequences, and hence tend to live their own life, independently of their creators, and mathematics as a human activity need its conventions, as in any linguistic activity. Mathematical Platonism essentially boils down to mathematical realism, and as noted the spirit of discovery has guided mathematician ever since, without it notions such as question of logical rigor would not be meaningful. In fact a real challenge to mathematical realism has only been mounted during the crisis of foundations during the early 20th century. The only serious contender to mathematical realism, whose roots, as so much in modern science and philosophy,ontological as well as ethical, can be traced back to Aristotle. This notion is that mathematics far from having an independent existence, is just a tool, a language in fact, to describe the real world, and thus potentially be replaceable. The notion of language should of course be taken metaphorically and not literally. When taken literally a metaphor ceases to stimulate the imagination and becomes merely silly.

Plato was not a mathematician, and his direct influence on mathematics has not been entirely beneficial. Supposedly his insistence that only ruler and compass should be used in geometrical constructions has rightly been criticized as far too restrictive. In fact, anti-Platonists have pointed this out as typical of Platonistic mathematics, and that subsequent developments of mathematics has been a release from constrained mathematical Platonism. Maybe a release from Plato, but not from Platonism. The Platonistic conception of mathematics has rather been vindicated than contradicted by the triumph of mathematics, especially as to its spectacular success in fundamental physics. In modern cosmology mathematics tends to be prior to the physical world, mathematical concepts somehow existing before physical objects. The spirit being turned into flesh at the Big Bang, a narrative that has very strong religious overtones. Which, for many people, may be a sufficient reason to be skeptical about it. More fruitfully Plato suggested a mathematical research program, essentially involving a development of three-dimensional geometry, with a view of describing the movements of celestial bodies. This would not take place in the time of Plato, but would occur only some two thousand years later, and eventually surpass anything that Plato concretely could have envisioned. After all he was but human.

Euclid and the Axiomatic Method

The great revolution of scientific thought due to the Greek was the introduction of deductive reasoning, both as a method to assure certainity but also to achieve clarity and transparency through the necessary intellectual hygiene of clear definitions and piecemeal reasoning. Euclid made a compendium of the geometrical knowledge available at the time, it is doubtful whether he in any way contributed any important new results, and besides the body of knowledge was very limited when seen in retrospect. But what he did was to present those results in a very nice and systematic way that has served as a model for natural science and mathematics ever since. For the first time really the pertinent features of an axiomatic presentations were spelled out. True, he had predecessors, Aristotle had presented a systematically deductive presentation of syllogisms somewhat before (Euclid and Aristotle did overlap biographically), but that task was much easier and much more formal, then the one confronting Euclid. Euclid did not want to formalize something already formal, but something physically palpable as the physical space in which we are embedded. As such it was not just a mathematical project it was a physical one.

Euclid first laid down the rules of the game so to speak. He gave a list of axioms and postulates, an important distinction which tends to be blurred today. Axioms involve principles of thought common to all logical thinking, while postulates concern self-evident facts about the subject matter, in this case physical (or mathematical?) space. He also tried to give clear definitions of the objects, such as points and lines. The first metaphysical insight is that one has to start from something, one cannot start from a void. Out of nothing, only nothing can ensue. Already by making some basic assumptions you make some concessions to arbitrariness, by admitting that some things cannot be proved but need to be taken on faith, and whatever that faith is based on, it cannot be deductive reasoning.

Now with the ground clear and all the cards shown, we can proceed piecemeal. The ambition is to make short steps and not to rely on anything but the basic assumptions and what can be derived from those deductively. In particular there should be no reference to drawn figures, which of course always tend to be particular. Now in this way Euclid proves a lots of facts, some auxiliary, referred to as lemmas, some central referred to as theorems, and some immediate consequences called corollaries. The general trend is to go from the simple to the more involved, gradually constructing an impressive edifice, in which every fact results from a sequence of compelling steps. But the presentation is not mindless, although deduction can be thought of as a mechanical process. Not all the facts are of equal importance, there is a structure in which some results are far more central than others. To single out the important theorems is something that goes beyond mere deduction. The achievement of Euclid is impressive. Much more goes on behind it than meets the eye. Still it is not perfect, as later generations of mathematicians would point out, there are many flaws.

First the definitions are far from satisfactory. Just as there are unproven statements there has by necessity to be undefined notions. If every notion has to be properly defined in terms of other notions, we either end up in an infinite regress, where the explanatory notions will be necessity become more and more complicated (as we run out of simple notions) or we will be caught in circularities. Neither of those stratagems is acceptable. Infinite regress means indefinite postponement, circularity means nonsense. In no case will there be anything of substance conveyed. The solution to this dilemma is to think of objects instrumentally. They are not defined explicitly, but implicitly in terms of the postulates, that allows us to manipulate them. This instrumental point of view was stressed by Hilbert. The objects undefined could be anything. The important thing was not what they were but how they interacted. Just as the pieces of chess has only meaning as to how to move them on the board, their actual shape and appearances being of no importance[12]. Now by the time of Hilbert the axiomatic approach had taken on a slightly different hue, and Hilbert went to pains to stress its purely formal aspects. Euclid was no formalist. To him and the Greek it was physical space that mattered, and lines and points had definite interpretations. Thus his circular definitions of the basic geometrical objects was to direct the intuition of the reader into the relevant furrows. Just as truth can be intuitive, so can definitions.

Now, especially from a modern point of view, postulates are neither true nor false in any metaphysical sense, just as little as the rules of a game are true and false, they are just conventions that define a game. However, Euclid was not concerned with a game, he was concerned with real physical space. His postulates about space, were not of the same type as the so called axioms of groups, but in the nature of self-evident truths about space. In modern axiomatics, the notion of self-evident has become irrelevant, not to Euclid. The truth of a statement, or rather the claiming of the truth of a statement becomes a moral stand. To say that something is an axiom involves a moral responsibility. This Euclid understood well. His notorius fifth postulate, on which all the deeper parts of Euclidean geometry depend, is a case in point. This postulate stands out, although intuitively obvious, or at least apparently so, it does not have the same simplicity and immediateness as the other postulates. It has the character of a theorem, of something deducible from simpler statements. It lacks the irreducibility that characterizes the others. It is tempting to assume that Euclid tried to prove it and eventually had to give up. To clearly state it as an unproven assumption was a testimony to intellectual honesty, which is of course a moral issue. In so doing, Euclid further clarified the nature of the axiomatic method, and by evading obfuscation (which a lesser mind may have succumbed to) he did generations of future mathematicians a great service.

Finally when we come to the axioms, we come to the heart of the matter, because axioms, in the initial distinction made between axioms and postulates, refer to principles of thought. Those are, unlike the postulates, inseparable from thinking. They are central to the very process of thinking, and thus something internal that cannot be externalized. The whole point of the deductive approach is to put the world under the power of our rational faculties. While postulates can be matter of convention, our thinking is not. The nature of the axioms are thus much more intimately than anything else in the character of being self-evident. Thinking itself is its sole justification. It is very hard to be aware of your thinking process, and in the process of deductive engagement it is easy to bring into play principles of reasoning which are not explicitly declared at the beginning. Many such tacit and implicit principles are used by Euclid, even those which may be more postulary in nature than axiomatic, such as metrical invariance of movement in space. The principle of superposition of triangles is repeatedly invoked. Or that a line segment joining an interior point of a circle with an exterior will actually intersect the circle. Thus although Euclid has the ambition to dismiss diagrams as anything more than as an imaginative support to the intuition, occasionally he subconsciously relies on them for unwarranted assumptions.

But flawed as Euclid may be under a closer scrutiny, the results he derives are still valid, which cannot be claimed, except for few obvious examples, for scientific claims of antiquity, and this alone is a noteworthy achievement. Furthermore the flaws of Euclid are not only understandable but inevitable. One may see the axiomatic presentation as a Platonic form, and any human representation of it is bound to be imperfect, just as the triangles we draw in the sand. Like all our human efforts they can be improved, but never made perfect. Hilbert's modification of Euclid is of course in the technical sense an improvement, but even modern axiomatic representations are bound to be flawed, because, as stressed, it is impossible to completely axiomatize all principles of thought, which makes the ambition of a universally formalizing mathematics an impossible one[13].

Euclid, although not perfect, has provided a lasting inspiration for mathematics, and also more generally for the scientific project, as we will see when we bring up Descartes. The axiomatic method has achieved even greater significance in the 20th century, as it is seen as a unifying strategy, doing multiple work simultaneously, by stripping matters of their inessentials, and high-lightening common structures. As such it has played an important role in the drive towards abstraction in mathematics, which has inspired in particular much of 20th century mathematics.

Notes

[1]The story of Linear B is fascinating, in the process of deciphering this strange script on clay-tablets, it turned out that the language was Greek, and thus the history of the Greek civilization was pushed back several hundred years. There had been, unsuspected by historians, a Greek Bronze Age Civilization associated to the archaeological finds at Mycenae, with a script. That Civilization perished and the memory of it seems to have been forgotten by later day Greeks, yet the language survived and the Greek were to flower a second time.

[2]To get a feel for an Ancient Greek temple you could do worse than visit a modern Hindi one in India with its gaudily painted figures and bustle. The classical Greek sculptures we today admire for their purity (and even disfigurement) may very well have been painted for enhanced likeness. Similarly with the white temple ruins. It is of course speculations, just to remind us about how much we do not know about the past as most of its traces have vanished.

[3]It also lies at the heart of democracy. Democracy is not about elections, ultimately it is about open discussions and the right as well as the necessity of scrutinizing arguments. This is the ideal way of arriving at a consensus.

[4]incidentally it was given to me as a child by my mother and for some reason the occasion has never faded from my memory

[5]With modern technology we have in some instances direct measurement of some close astronomical distances within the solar system. Those are based on the velocity of light, assumed to be constant throughout space. Note though that the first estimates of that velocity was done in the 17th century by the Dane Rømer, using the Doppler effect of the Galilean satellites, and thus being based on geometrically based distances. Modern calculations are confined to the earth.

[6]This reminds me of a grandfather of mine who was a farmer and for the sake of construction of some machinery systematically measured the circumferences and diameters of wheels of different diameters. and thus chanced upon the number π. Thus similarity is not something which is innate, but something we discover.

[7]True, the Greeks did not have access to modern algebraic notation, but it does not take too much to convince you that the product of two odd numbers is an odd number by a graphical representation if nothing else. But compare the remarks below.

[8]More exactly. 'Deduction compels, induction permits' Collingwood (1889-1943) was known for his opposition to the more and more technical emphasis on philosophy due to the school of Analytic philsophy developing primarily in the Anglo-Saxon World in the 19th century, although many philosophers think of it as the true heir of the great philosophers of the past, thus part of the main tradition. Collingwood, whose ideas about history are fascinating, was no wooly-minded thinker and had also incisve things to say about logic.

[9]This means of course that $\frac{1}{x} = \frac{x-1}{1}$ which translates into $x^2 - x - 1 = 0$ with the solutions $x = \frac{1 \pm \sqrt{5}}{2}$ the bigger solution corresponding to our case.

[10]The number x has, as the Eucldan algorithm shows you, the continued fraction expansion $1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\dots}}}$.

[11]nevertheless there has been a steady stream of attempted solutions sent in by amateurs of trisecting an angle. Thus a trisector has become synonymous with a crank, the second problem of constructing the cube-root of two does not seem to have inspired a similar flurry of activities.

[12]Of course if you want to physically manifest the game of chess there will be constraints on the physical properties of the pieces. They have to fit on the squares, and stably so, not toppling over, nor evaporating during the time it will take to play the game. But this is just a concession to interior decoration and fetishism, serious players of chess supposedly have no problems with so called blind chess.

[13]There are of course other obstacles to axiomatization as a means of achieving certainty, due to the impossibility of proving consistency, to which we will return.