Eleventh Lecture

Foundations of Mathematics

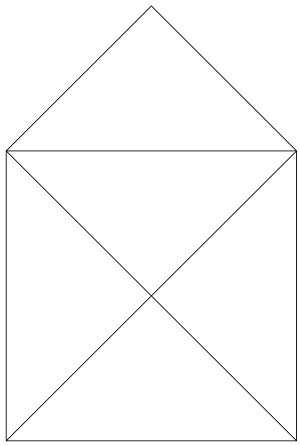
> *Mathematics is the Science which draws necessary Conclusions, not the Science of necessary Conclusions* (Benjamin and Charles-Saunders Pierce)

A mathematician engages in logical reasoning as naturally as the speaker speaks without being in the least conscious of the physiology of vowel formations and such things. In a way our reasoning power is innate, we normally do not question its principles, we take them for granted. Logic is the attempt to scrutinize those principles to see whether they are sound or not. But of course the project seems doomed to a viscious circle. On what grounds do we judge our reasoning powers, if not by our reasoning powers? The project appears indeed doomed, on the other hand we are all doomed to ultimate death, and that means all of us, but we do not give up and die just because of that, but take a more pragmatic attitude. In the same way we can explore the issue and see where it might lead. This is called curiosity, and is the driving force of us getting up out of bed every morning. So the question is to what extent is mathematics based on logic? Does logic provide the foundations of mathematics? More interestingly though is to what extent is mathematical reasoning not only constrained by logic but indeed driven by it? This means, can mathematics be pursued mechanically, it only being the case of following logical rules, and if so where does creativity enter? Russell and Wittgenstein claimed that mathematics was just a matter of stringing tautologies one after the other. Although both were mathematically literate, Russell probably more so than Wittgenstein, neither of them had done any creative work in mathematics. C.S. Pierce, quoted above, held a different view. To him the natural integers were more basic than logic. Thus logic needed an infusion of mathematics to be interesting and rise above the classical level of Aristotle, more than mathematics needed logic to justify its conclusions.

We started the whole story with the introduction by the Greeks of deductive reasoning in mathematics. As I pointed out the major point of deduction is not to attain unquestionable certitude, this is an elusive quest, but to make your reasoning transparent and hence open to criticism and modification to lead to improvement. This is the basis of democracy namely an open discussion and in which arguments are considered on their own merits regardless of who propose them. In any deductive reasoning you have to state your basic assumptions, not all of them, because you will never be aware of them all, only those that may be controversial, and also controversies cannot be predicted from the outset but will emerge as the discussion proceeds. If the discussion leads to absurdities or actual contradictions it will be an occasion to reconsider basic and often tacit assumptions. This is also the case for science in general, although this does not lend itself so easily to clean deduction.

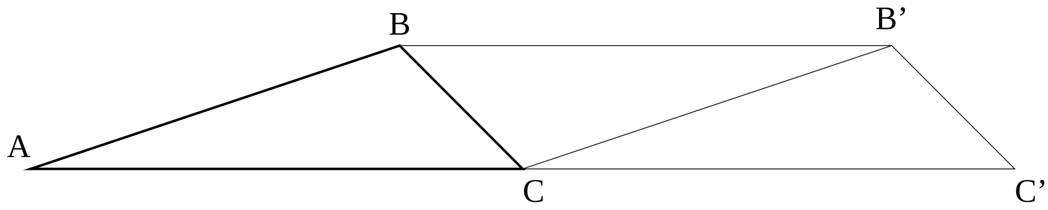
The case in point is the axiomatization of geometry, where the Greeks did not envision a purely mathematical project but more of a physical, describing the real physical space around us, be it in an idealized way, meaning using ideal tools such as straight lines, indefinitely extended without any thickness. As we noted initially with Thales, without an axiomated setting it is not so easy to 'prove' geometrical statements, what should you take for granted, what should you derive by reason? And even intuitive things may also be based on reasoning of some sorts, be it subconscious.

As a further illustration we may consider some examples of simple and striking reasoning which, according to Popper, need not be based on axioms. One example is that the area of a square with the side equal to the diagonal of another is twice that of the latter. According to Popper, referring to Plato, the following figure proofs it at one glance without the need of an axiomatic machinery.



Another example is taking a triangle and sliding it along the line AC' that extends its base AC. Popper takes for granted that the triangle $BB'C$ which is formed is the original ABC only translated and flipped upside down. From this we see immediately that the angular sum of a triangle is the same as the angle of a straight line (or half a circle).

The arguments are of course compelling and induce you to make cardboard copies and demonstrate physically. But as noted in the sensous domain there are no straight lines, and cardboard templates are crude indeed, those experiments can only be done in your head.



Now the ambition of the deductive approach was to make the proofs independent of visual confirmation, as opposed to mere guidance. As geometry is an eminently visual field, as these proofs testify, the ambition is hard to uphold, and Euclid repeatedly sins against it. In fact the very visuality and palpability of geometry makes it rather resistant to a formal axiomatization, as you will invariably resort to principles of reasoning which have a strong visual and even mechanical character, (and recall that much of our familiarity with space is not visual but based on our ability to move in it and handle its objects, as we are indeed invited to do in those pictures), making them hard to clothe into words. The fractal mathematician Mandlebrot claims that there are two kinds of mathematicians, those of the eye, and those of the ear. It is hard to translate geometry into music. The ability of Euclid to nevertheless achieve such a successful translation is a testimony to his genius, or at least the majesty of his accomplishment, which is far from trivial, and has had a profound impact on the future of mathematics.

In retrospect we note all the arguments are based on the tacit assumption of the parallel postulate. To recognize it is not trivial, but a consequence of the deductive approach, whose main merit is its transparency. To recognize it, is to discover it, and once it has been discovered to contemplate its possible negation. This illustrates the power of the deductive approach when it comes to scrutinizing assumptions and hence as a tool of discovery. Hyperbolic geometry may never have seen the light of day otherwise, as this is not, unlike spherical geometry, naturally part of Euclidean geometry[1].

The human mind has limited powers of cognition and in fact many mammals and even birds may be superior than humans in many respects. In fact the human ability to subitize, i.e. to tell the number of objects in one glance without counting seems not to be better than those exercised by rats and ravens, maybe even less. The process of counting is a way of transcending those purely biological constraints of man. In the same way mathematics in general provides a means of transcending our imagination. One example may be 4-dimensional geometry. It would be impossible to conceive of it consistently if there had not been the invention of coordinate geometry which involved what later would be called the arithmetization of geometry. Once this is in place, the formal extension to four dimensions (or any number of dimensions) is trivial and almost immediate. But when it comes to four dimensions our human visual cognition deserts us (in spite of some heroic efforts) and hence our palpable intuition, yet there is for us a world ready for exploration which we would never have normally thought of [2].

The Classical Greeks were very concerned with rigor which had some advantages as well as disadvantages.

The advantages we have already discussed earlier. The tradition of rigorous reasoning, a tradition which has only started once in the history of mankind, which arose among the Greeks, lifted mathematics to a new level, and we can even claim that it really meant the birth of mathematics as we know it. This tradition is still very much alive, and makes it possible for us, in spite of a gap of two millenia (and more) to read and appreciate their texts, which is not possible without great effort and sympathy, with the texts, often younger, of other mathematical traditions, which have at best only marginally contributed to mathematics. That does of course not mean that the achievments of other traditions are worthless, they have their interests, but more of a historical nature showing a range of alternatives and shedding some light on the nature of mathematics as a human endeavor, in particular its basic universality.

There are also disadvantages of rigor, or at least of an excessive one. Rigor as a guide to truth is one thing, then it serves a purpose which best can be described as moral. But rigor can also ossify along formal lines and thus impede discovery rather than abet it, and become a straight-jacket, as with all traditions when taken literally rather than following the spirit. As a concrete example one may mention the disinclination of the Greeks to handle real numbers uniformly, and instead pursuing the logically impeccable but cumbersome method of proportionalities, which foreshadowed the method of Dedekind, as we will encounter below. The problem is that the Greeks thought of quantities, and they come in different kinds, just as in physics, where you cannot mix m with m^2. Thus to the Greeks the multiplication of two numbers, say related to length, had to be interpreted as an area, and thus the multiplication of four numbers did not make geometrical sense. The indiscriminate use of numbers in addition and multiplication is somewhat confusing and involves a departure into what potentially could be quite bewildering. Basically it entails the abandonment of geometric visualization for formal manipulation which is instinctively anathema to a mind wedded to a tradition of clear thinking. This grouping in the dark would be typical for the early European tradition as we have noted, and one may also think of it as the source of the familiar saying that mathematics is not something you understand but get used to.

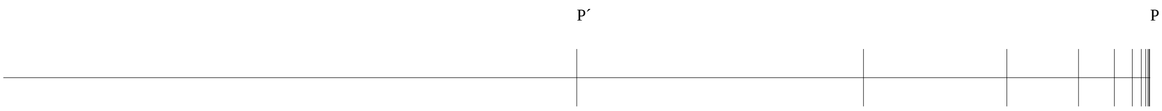
Another criticism of Greek mathematics was its purity, meaning that it was adhering too strictly to the Platonic ideal of unassailable truth, thus disparaging its mundane use in practical situations, where idealism may have to take second place to pragmatism. Now the point of applied mathematics is not its usefulness, at least not from the perspective of a mathematician, but its extending of horizons, i.e. to widen the field of mathematical inquiry and thus launch it on its way to discovery of new lands, just as the human exploration of our planet may have been ultimately urged by curiosity not commercial interest. What is the most exciting theorem in Euclid? There are many nice facts exposed but maybe the Pythogarean one is the most striking, unpredictable as it is. Symptomatically it also predates Euclid, to whose credit belongs not discovery but beautiful and economical confirmation. Taken as a whole the subject matter of geometry as presented by Euclid is rather limited. One may note that in Hellenistic mathematics there were some new lands opening up as new avenues were being explored. A trivial example would be the formula of Heron giving the area of a triangle given the lengths of its sides. As triangles with the same sides are congruent, the area should in principle be obtained from the sides, but classical Greek geometry was not interested in any such explicit formula. From a practical point of view it allowed the enclosed area of a triangular region to be computed without entering it. Symptomatically Archimedes made a distinction of methods used for discovery and those more demanding one for proofs. As noted earlier the rigor of his presentation surpassed those of Newton and Leibniz and are still accepted today.

Now when mathematics revived in the 16th century there were no longer the same constraining regard as to rigor. First new tools appeared foremost of which was a much more extensive use of algebra and algebraic manipulation which was extended to the infinite realm with the invention of calculus (which were originally thought of as algebra). This was the introduction of formal methods, which were not secured by any axiomatic basis. In a way this was not so important, algebra proceeded along certain rules to which you had to adhere, thus you could not just make up algebraic expressions at random, you had to reach them by adhering to strict rules. And as I never tire of repeating, your imagination is only stimulated by obstacles to overcome, and it becomes doubly important when you explore the creations of your imagination to be subjected to unforgiving rules. Thus the rules in a sense served the same purpose as rigor, as a guide to discovery. And as to the soundness of the rules, that was not an issue because as long as the results seemed reasonable there was permission to use them without bothering too much whether their use was justified on first principles.

As an example of discovery through formal manipulations one may point to the discovery of imaginary numbers in connection with Cardano's formulas for the solution of the cubic equation. The noteworthy thing was those came into play when there were three real roots, which were presented in this rather round about way. At a time when negative numbers were viewed with suspicion as a result of associating numbers with either cardinalities or quantities, square roots of negative numbers were doubly suspicious, but if the manipulation of formulas forced them on you, you better accept them, at least provisionally. Thus algebraic manipulations allowed you to navigate in a world of the imagination that lay beyond the extension of your natural biological cognition.

Now the algebraic manipulation of formulas was one thing, the extension into infinity was another thing, as long as there were no rigorous definition of taking limits. But the limit concept was rather intuitive and here again practice condoned, as most of mathematical work was geared towards applications to nature, and hence nature itself could be used as a check. It was different with purely mathematical realms such as number theory, where there were few if any rigoros demonstrations at the time. A result in number theory should hold for all integers, and unlike nature there are no representative examples. Results in number theory can of course be checked for small numbers, but that is only a way of catching obvious mistakes, there is no gaurantee that it will work for big ones. There is no such thing as a typical number, all are too small and belong to a minority. As far as there were proofs those were based on general almost meta-physical principles about numbers, such that there cannot be an infinite descent (a principle much used by Fermat), which is of course a reformulation of the inductive principle, or equivalently the well-ordering of the integers, to the effect that every non-mepyy subset has a smallest element[3].

In the beginning of the 19th century there was a movement for rigor in analysis. To introduce it was not trivial, just as with geometry in classical times. Abel and Gauss belonged to the pioneer. Gauss, as always were in the forefront anticipating not only his contemporaries, but also his more recent predecessors. Abel claimed that there were no single infinite series in analysis whose convergence had been proved rigorously. This may be somewhat of an hyperbole considering the geometric series known since antiquity, yet he had a point. From a logical point of view the difficulty is that you cannot tell whether a particular series converges or not just by looking at a finite number of terms in the beginning. To prove convergence you need to consider in your mind an infinite number of cases. As this is impossible in a physical way you may ask what are you doing in your imagination. As an example consider the sum of the geometric series $\sum_{n\geq 0} 2^{-n}$. There is a geometric way to see it, reminiscent of Zenos paradox.



We see how the initial partial sums P' approach the end point P at 2 by successively halfing what is left from P' to P. In our mind we can perform this operation a dozen time and hence get the feeling that we can do it indefinitely, as each step basically leaves it in the same situation as before, just the case of induction from n to $n+1$. in particular through Cauchy and later Weierstrass there was an exact definition of limits using ϵ's and δ's

There was rigor in showing that continuous functions attain maxima and minima on compact sets (closed bounded intervals) following Bolzano, and intermediate values. This presupposed a rigorous definition of what is a real number which only appeared in the 19th century. There were different approaches such as the cuts of Dedekind and convergent sequences (Cauchy sequences) by Cantor, all of them entailing a completion of the rational numbers, based on very compelling intuition. So let us digress.

How do we know that there is a number such as $\sqrt{2}$? As we know there cannot be a rational number x that satisfies $x^2 = 2$ but there is a segment, or rather there are two segments whose proportion satisfies this, namely the side of a square and its diagonal. Using the formalism of Eudoxus as explicated in Euclid you come up with Dedekinds cuts. Explicitly if the real numbers are partitioned into two sets X, Y such that if $x \in X, y \in Y$ then $x < y$, a so called cut, there is a number c such that $Y = \{y : y > c\}$ or $Y = \{y : y \geq c\}$. As an example we may consider $X = \{x : x < 0 \vee x^2 < 2\}$ and $Y = \{y : y > 0 \wedge y^2 \geq 2\}$ where c then would correspond to $\sqrt{2}$. A more direct approach would to consider Cauchy sequences of rational numbers, and posit that they are all convergent, or equivalently identify real numbers with such Cauchy sequences with a suitable equivalence relation.

> A sequence (x_n) is said to be a Cauchy sequence if for every $\epsilon > 0$ there is a number N such that if $m, n > N$ then $|x_n - x_m| < \epsilon$. Furthermore we say that two sequences x_n, y_n are equivalent if for any $\epsilon > 0$ there is N such that if $n > N$ we have $|x_n - y_n| < \epsilon$. It is easy to see that this is an equivalence relation (note reflexitivity pressuposes that the sequnce is Cauchy) and we may easily define the arithmetic operations on them by doing them pointwise[4] and an ordering by $(x_n) > (y_n)$ if there is an N such that if $n > N$ we have $x_n > y_n$. We can then, in the words of logicians, present a structure which is a model for the axioms of the real numbers. More down to earth we may also identify the real numbers with unending decimal expansions, or more elegantly by unending dyadic such, which is intuitively much more accessible and straightforward, but of course equivalent with what we have above.

This completeness of the reals can be expressed in many ways, one very convenient one is to consider an infinite set of closed bounded intervals I_n nested in each other, i.e. $I_n \subset I_{n+1}$ then there is a point p such that $p \in \bigcap I_n$. Note that p is unique iff the lengths of the I_n tends to zero. We can now prove many intuitively obvious facts that a continous function is bounded on a bounded closed interval and that it attains its maximum and minimum and every value between them. As illustrations we digress on it below.

> Let f be a continuous function on an interval I. Divide the interval in two (say by halving it) then f must be unbounded on at least one of them, if on both pick say the left. We then get a sequence of nested intervals and a point p in them all. By continuity at p we find that for some interval

J around p the function f is bounded (in fac there is some interval of length 2δ such that f is bounded by $f(p)+\epsilon$ using the standard definition of continuity. On the other hand there will be intervals I_n in the nested sequence on which f is unbounded and which are contained in J.

In the same vein, we can show that if f changes sign on an interval, it will attain the value 0 somewhere (and by letting a be any intermediate value between max and min, we can look at $f - a$ instead). Say if $f(a) > 0$ and $f(b) < 0$ and choose c halfway between say. Then either $f(c) > 0$ or $f(c) < 0$ (if $f(c) = 0$ we are done, or if we structure the proof differently, we can assume $f(c)$ never zero and proceed to obtain a contradiction). In the first case we look at the interval $[c, b]$ in the second case the interval $[a, c]$. And then we proceed, obtaining nested intervals in which f changes signs. At the common intersection point p we note that in a small neighbourhood J of p the function does not change sign because of continuity and the assumption $f(p) \neq 0$, getting a contradiction. We also note that the procedure we use can actually be implemented to find a zero, although it is not the most efficient, but will always work[5]

Now you may have heard about the expression. The axioms of a group. It sounds like an oxymoric. We are used to the notion that axioms are self-evident truths which we cannot doubt. In what sense can we doubt the axioms of a group? They are not axioms as much as rules. Rules cannot be doubted because by fiat they are true. No one in his right mind would doubt the rules of chess, because if we change them, we are no longer speaking about chess. Those axioms are simply conventions, they state the rules of the game. In Euclidean geometry, geometry came first and the axiomatization was a *a posteriori* formal justification. In fact geometry can be axiomatized in many different ways, and it is a rather difficult question how to axiomatize it in a useful way. The very fact it can so be done was a feat, perhaps not properly appreciated. In facts attempts have been made to axiomatize quantum mechanics, thereby reducing it to a mathematical game, but those attempts have so far, as far as I know at least, not been very successful. Now groups existed before they were defined, and initially they were thought of groups of permutations, as in Galois theory. But to axiomatize groups is not that difficult, although not entirely obvious, except in retrospect. There are many ways in principle in which you could generalize the concept of a permutation group, leading *a priori* to slightly different concepts[6]. Anyway when it comes to algebra, the formal rule bound structure of algebra makes it very amenable to axiomatization, and the process takes on an entirely different meaning. Rather than trying to describe an existing reality it creates one by convention, just as we make up a game. In the first case we have a reality, whose importance we have no reason to doubt, in the second case we create a reality, a toy-reality, whose importance and interest may be negligible. In fact we may not even know if it exists, meaning whether it is consistent. However, that question is about reality. The axioms and objects they regulate may not relate to anything at all. Euclidean axiomatics relates to space, and even if it is impossible to physically represent perfect straight lines with no thickness, the question of their existence in some other mode remains, and feels very tangible. One way of showing existence is to create structures which satisfy the axioms, one speaks about models. If a model has been exhibited one may with compelling justification claim that the system is consistent, that it describes things which exist. However, the structure that is created maybe very abstract by itself and build on some tacit system of assumed axioms. Examples of such are the non-euclidean geometries. Models of those can be created using concepts from euclidean geometry. The axioms of two-dimensional spherical geometry are easily seen to be satisfied on a sphere in Euclidean 3-space, with suitable definition of a line (as a great circle), while that of a hyperbolic plane, with lines given as circular arcs orthogonal to the boundary of a circle, is a bit more subtle and unlike the previous remained undiscovered until the end of the 19th century. However, the Greeks did not think of the sphere as representing a different kind of geometry, there was only one geometry, a statement which can be thought of as moral in the very same sense as there is only one God, or one reality. And besides a line is a line, it is straight and not curved like a great circle on a sphere. The instrumental notion that objects are implicitly defined by their relations to other objects, and have no other content, only slowly emerged, and got its first clear formulation by Hilbert in his attempt to make a logically impeccable updating of the Euclidean axioms (in other words trying to bring into the open all the hidden assumptions). However, such an ambition was implicit in the motivation of Euclid, to present arguments which did no rely on visual representations, but solely on the logical content of the objects. As Berkeley never tired to point out, you cannot draw a general triangle, only specific ones. A logical argument on the other hand, applies to a generality and is not tied to a specificity. However, as human beings we do prefer pictures, because even if they are specific they can nevertheless serve as representations, and we are, or at least we believe we are capable of abstracting from a picture the essential thing and disregard all the other aspects. Lagrange was very proud to have produced a book on mechanics without a single diagram, thus achieving a logically self-contained account. Thus the ideology of formalism has a long pedigree. Axiomatics have many advantages, which it shares with algebra, namely a great economy, and a release from thinking. The true nature of an object does not enter, only the way it behaves under manipulations. Furthermore an axiomatic treatment can unify diverse areas and results have a far wider application and relevance than if they had been confined to a single territory. Thus the axiomatic method made great stride, especially in subjects such as algebra, which would more and more be thought of as the science of structures. Emmy Noether effected a revolution with her 'abstract algebra' (the qualifying adjective was soon dropped) an approach which would have a profound influence on 20th century mathematics[7]. In this context one may also refer to Hilbert's celebrated proof of the finitely generatedness of the rings of invariants, which was effected by a simple but abstract argument (having to do with what later would be called Noetherian rings and their modules). While the classical invariant theorists were concerned with explicit generators, Hilbert sidestepped the entire issue, making the nestor of the subject - Paul Gordan - exclaim 'Das ist nicht Mathematik, Das ist Theologie'[8]. Maybe not that unapt (if inept). The proof was done in the modern instrumental view, inspired by axiomatization, looking for the logical essence of a mathematical concept, not so much as its manifested, which could be thought of as sentimental. The classical invariant theorists loved their formulas, and Hilbert's approach was in no help in finding explicit bases, only to show that they existed, which was after all the question of the conjecture, known at the time as Gordans conjecture. Thus one may speak about a split among mathematicians, (although of course any partition of mathematicians into disjoint groupings is moribund, as the groups will overlap extensively in practice), the concrete sentimentalists, who cherish the objects by themselves, and the more hardcore strategists, that dispense with the particulars and look for the essential approach (Grothendieck would be the example par excellence).

Mathematics took another turn at the end of the 19th century, and I am thinking of Cantor and his hierarchy of cardinalities. Cantor started out as a regular mathematician so to speak, concerned with problems in Fourier analysis. When you get into the mathematical details of the subject, you enter into some rather subtle questions concerning the real line, which you think of as having an almost physical reality. Cantor asked that If you take away the isolated points of a set and continue, will you have a set without isolated points? When you take away all the isolated points, and if there is an infinite number of those, there may still be isolated points left, even if they were not isolated to start out with. The typical example being the example above with $2, 2 - 2^{-n}$, where 2 is the only non-isolated point, but becomes isolated after all the isolated points have been removed. This leads to ordinals. After counting $1, 2, 3 \ldots$ there is still one more point to pick call it ω. Then you can start counting again $\omega + 1, \omega + 2, \omega + 3 \ldots$ to $\omega + \omega = 2\omega$. And you will continue, encountering $3\omega, 4\omega \ldots \omega^2, \ldots \omega^3 \ldots \omega^\omega \ldots \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}$ ... Will it ever end?. What we have here is a rather complicated set which we cannot really always expect to imagine, and it is defined only through a long process which we may never go through. [9]. Thus through considerations as such Cantor came to the notion of sets, and that those sets were only interesting when they were complicated and defied the imagination. Nowadays, set theory has become the language of mathematics, and thus we are at an early age introduced to rather trivial sets, such as the set of solutions to an equation, or the set of children to a mother, notions which have been clearly understood from time immemorial. In elementary mathematics little is gained by formalizing such trivial notions, and probably much confusion results instead. Now the really interesting notion of sets is that of cardinality, and the surprising discovery is that there are different orders of infinity, in particular there are uncountable sets, sets for which you cannot pick up the elements one by one, and eventually pick them all. The uncountability of the reals lies at the heart of its essence, and may have been a notion that the Old Greeks may have chanced upon, although admittedly it lies deeper than the paradoxes presented by Zeno. It is very elementary though, but I suspect few people not of a mathematical temperament can understand it and appreciate it properly[10]. It is based on a simple idea which permeates much of modern logic. The argument is well-known, and many you may have encountered it as teenagers[11], but let us repeat it.

The usual way it is presented is to look at infinite decimal expansions. And given a list indexed by the integers to create a decimal expansion that differs from all those appearing in the list by at the n-th expansion change the n-th digit in the expansion. This expansion will differ from the n-th at the nth digit and hence not be identical with it.

You see that it is so simple that it can be expressed in ordinary language without anything getting lost. And the idea is so simple that once you have encountered it you cannot forget it. Often it is accompanied by a picture, something like this, which if you have not caught on immediately, certainly will do so now. Note the diagonal that represents itself, hence the argument is often referred to as Cantor's diagonal trick.

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0.91182080351557135663823489021468788623220470879045
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You can of course express it using mathematical logical formalism. Given a function $F(m,n)$ from the $\mathbb{N} \times \mathbb{N}$ to $\{0, 1, ..9\}$ consider $\psi(n) = F(n,n) + 1$ where we should think of addition as modulo ten. Then there is no m such that for all $n \quad F(m,n) = \psi(n)$. Or even more formally $\forall m \exists n \neg(F(m,n) = \psi(n))$. Essentially, in an isolated case like this, it adds nothing, and remains nothing but pointless pedantry. The precise statement is a tautology, its meaning has to be explained. It may be considerations like this that made Russell (followed by Wittgenstein) to think of mathematics as sequences of tautologies.

Now there are certain objections which could be made. One is that the decimal expansion of a number is not unique, that those that end with zeroes can as well made to end by an unbroken sequence of 9's instead. It is sightly annoying, does not invalidate the idea, and can easily be taken care of. A more common objection is that the missing expansion can easily be added to the list (by being put first and all the others shifted by one). Any number may of course be listed, and every list can be extended. The point is that any list will by its very existence give rise to a missing expansion, so if we would try to extend our list indefinitely by always adding no expansion, the process would never stop in the sense that no complete and final list would arise.

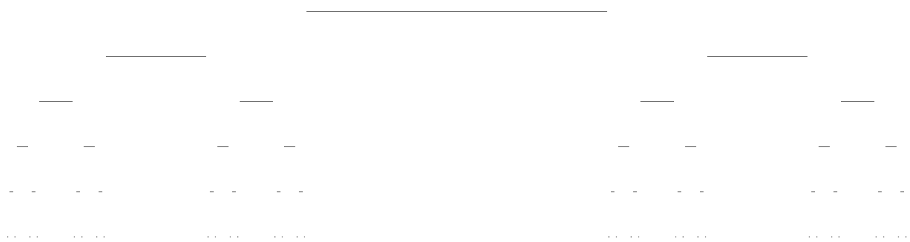
Now this has a lot of consequences. Do transcendental numbers exist? Liouville had come up with the clever explicit $\sum_{n>0} 10^{-n!} = 0.110001000000000000000001\ldots$ the idea being that it had too close approximations by rational numbers. In fact if θ is a solution to a polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \ldots a_0 = 0$ with integral coefficients and $\frac{p}{q}$ a rational approximation then $|a_n(\frac{p}{q})^n + a_{n-1}(\frac{p}{q})^{n-1} + \ldots a_0| \geq \frac{1}{q^n}$ from which follows that $|\theta - \frac{p}{q}| > A\frac{1}{q^n}$ where A is a universal constant only dependent on θ not on q. Cantor showed that in fact most real numbers are transcendental, while the algebraic numbers are countable[12].

Now with the notion of uncountability we may return to Cantor's original problem. Say that a point p is a condensation point of a set X if every neighborhood of p contains an uncountable number of elements of X. We may consider the subset $C(X) \subset X$ of all the condensation points of X and it will turn out that $C(C(X)) = C(X)$[13], thus $C(X)$ will be untouched by the process of picking isolated points of X to remove them.

This notion leads to the concept of a perfect set, meaning a closed set without any isolated points. Such a set is by necessity uncountable unless empty. If you start with a closed set and remove all the isolated points, the set will remain closed. If you do this process indefinitely, you will have a sequence of nested closed sets, whose intersection will be closed, hence in the end you will end up with a perfect set. Thus if we have a closed countable set, we may indeed reduce it to an empty set by the recursive process.

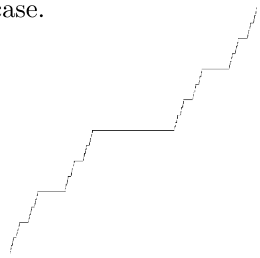
A famous example of a perfect set with no interior is the Cantor set. It is defined inductively by dividing up the until interval into three parts and remove the middle $x : \frac{1}{3} < x < \frac{2}{3}$. This leaves two closed intervals on either side and submit them to the same treatment, and proceed inductively. We will end up with a set C such that $C = \frac{1}{3}C \cup (\frac{1}{3}C + \frac{2}{3})$ or equivalently we are looking at numbers in the unit interval, whose triadic expansions contain no 1's. This is really the first explicit example of a fractal set, with its simple self-similarity. Note that the self-similarity shows that its Lebesgue measure $\mu(C)$ has to be zero, as it satisfies $\mu(C) = \frac{1}{3}\mu(C) + \frac{1}{3}\mu(C) = \frac{2}{3}\mu(C)$[14]

We see that we encounter a 'new kind of mathematics', dealing with a new kind of idealized objects, necessitating infinite processes, which we in our minds must follow to the very end. Just as it is impossible to physically draw an infinite line, infinitely thin; it is impossible to draw the Cantor set, but that should not stop us from trying. First the five first steps in the process of removal



And then the result of the five removals

If the process would be allowed to go to infinity, it would of course be invisible. There is also another way of representing it, through the so called Devils staircase.



This gives a continuous function which is constant on the complement of the Cantor set, and hence has derivative zero there, but is not differentiable on the Cantor set, the difference quotient going to infinity. It maps a set of measure zero to the entire unit interval.

Now the uncountability of the reals can be thought of as a metaphysical statement. A fact about the existence of an infinity beyond that of the ordinary infinity as represented by the unending succession of integers. It thus pertains to the mystery of the continuum, which perplexed and fascinated Zeno. Now it is a consequence of some rather strong intuitive properties of the continuum, which might have been readily accepted by the Classical Greek culture, such as the intersection of nested bounded closed intervals being non-empty (the fact that it can be empty as in the case of the rationals certainly would have been more apt to surprise them. But does it have any real mathematical content?

As noted the ideas of Cantor fits into a general inquiry about the deeper nature of subsets and functions, which was never undertaken before. The Devils staircase is a function which is not readily given by a formula[15] nor is it possible, as little as the Cantor set, to represent it faithfully as a picture. One may ask if this is just frivolous play? In what sense do these idealized constructions have a real existence? Are they just pathologies, or as Mandelbroit, later would claim, represent the typical? It is at this time Weierstraß exhibits continuous functions, non-differentiable everywhere, and Peano constructs space-filling curves. All those constructions being limits plied to the very limit of perfectly reasonable objects. As the notion of limits become more exact and rigorous, it also serves as an inspiration to the imagination to think about the unthinkable, which is one of the hall-marks of mathematics. It is at this time it becomes important to find measures which are countably additive, and notions of integrals such as we may have $\lim_{n\to\infty}\int f_n = \int \lim_{n\to\infty} f_n$ which led to Lebesgue measure and Lebesgue integrals, necessitating considering rather complicated, and in a sense non-mathematical sets. Now in this context, uncountability is essential, would the reals be countable, and yet every point having measure, the reals would have measure zero. It is a trivial remark, but it shows that rigorous analysis cannot exist logically without the notion of uncountability. Philosophically one can put it this way. The uncountability of the reals allows them to provide the context in which to freely exercise countably infinite processes, which lies at the heart of hard analysis, whose position is thereby somewhat paradoxical. On one hand it ties in with reality through the classical connection with physics, one example being Fourier series; on the other hand it becomes a hard exercise in a ghostlike world of daring infinite processes. The subtle questions about convergence of Fourier series is an example where physically motivated questions leads into very subtle terrain, which has no physical relevance at all.

Now to return to Cantor. Having shown that there are two kind of cardinalities, the natural question is whether there are more. $\mathbb{R}^2$ should contain more points than $\mathbb{R}$. It turns out with a simple argument that this is not true. No finite power $\mathbb{R}^n$ of the reals have more points than the reals themselves. This even holds for a countably $\mathbb{R}^\omega$! One should contrast this with the powers of the set $\{0,1\}$ whose finite powers all are finite, while its infinite power $\{0,1\}^\omega$ is uncountable. With a suitable topology on the product it can be shown to be homeomorphic to the Cantor set![16] It can also be thought of as the set of maps from the integers into $\{0,1\}$ and as such as the set of subsets of the integers, whose uncountability can be exhibited just as above with the Cantor diagonal trick, in fact it being easier as there are no technical difficulties with ambiguity. In fact the same argument can be applied to a far more general situation. Specifically, given a set X we can form its power set $\mathcal{P}(X)$ as the set of all its subsets. The point is that X and $\mathcal{P}(X)$ cannot have the same cardinality, because if there would be a $1:1$ correspondence $\Phi : X \to \mathcal{P}(X)$ we can form the subset $Y \in \mathcal{P}(X)$ defined by $y : y \notin \Phi(y)$. Now let $x \mapsto Y = \Phi(x)$. By definition of Y we have that if $x \notin \Phi(x)$ then $x \in Y = \Phi(x)$, and if $x \in \Phi(x)$ then $x \in Y = \Phi(X)$. In both ways we get a contradiction through the assumption of the existence of Φ hence we want to discard it. To make the connection with the original diagonal argument more clear, we can think of a subset Y of X as a sequence (a_x) of 0's and 1's indexed by the elements $x \in X$. Thus we have $a_x = 1$ iff $x \in Y$ and $a_x = 0$ iff $\notin Y$. Thus we consider for each x the sequence $(\Phi(x)_y)$ indexed by the elements $y \in X$, and define a new sequence (a_x) by changing the x component $(\Phi(x)_x)$ or $a_x = 1 - \Phi(x)_x$. Thus we have $a_x = 1$ iff $x \notin \Phi(X)$ and $a_x = 0$ iff $x \in \Phi(x)$. Thus the subset corresponding to (a_x) will be exactly the subset we defined above.

Thus we have an infinite sequence of higher and higher cardinalities, by considering inductively $\mathcal{P}^{n+1} = \mathcal{P}(\mathcal{P}^n(X))$ with say $\mathcal{P}^0(X) = X$. And we can do one better by considering $\mathcal{P}^\omega(X) = \bigcup_n \mathcal{P}^n(X)$, and after that $\mathcal{P}^{\omega+1} = \mathcal{P}(\mathcal{P}^\omega(X))$ and we are back to the ordinals again. And when will this end?

There is an ultimate set, the set of everything, or at least the set of all sets.

We can think of that as the universe $\mathcal{U}$, and clearly there can be no bigger set than this, thus it will have the highest cardinality, certainly not $\mathcal{P}(\mathcal{U})$. What goes wrong? Go back to the proof of Cantor and come up with the following definition;

Let X be the set of all sets which are not members of themselves

Is X a member of itself? Then $X \notin X$. If X is a member of itself, it is by definition not a member of itself.

This argument most of you have encountered before. It is known as the Russell paradox, which did away with so called naive set theory.

According to naive set theory you can come up with any kind of property and consider the set of all objects with that property. Conversely given a set, it provides a unifying property for all its members, by virtue of being a member of the set. So naively you think that there might be many more sets than you can come up with properties for. That the general set is somehow an arbitrary collection of elements that you have simply put together without any formulated property[17], and that there will be far more sets than you will ever have the time and inclination to form, just as there are far more numbers out there than you will ever encounter, and as to sets, even more than ever, as the number of subsets of the integers is uncountable. Could it be that they are only known to God? (His main activity and rationale for his own existence?) why for humans they only come into existence as we look at them?[18].

Another example of a naive way of reasoning was what was later would be labeled as the axiom of choice. Is it possible to out of a set of sets, pick one element from each set and make a set of it? In our everyday world of experience this seems a pretty harmless activity, but what about in mathematics? Define an equivalence relation on the reals by saying that two real numbers are the same if they differ by a rational. In this way we partition the reals into a uncountable number of countable equivalence classes. Is there a set which contains one member from each? Such a set needs to be uncountable, but can we really construct it, or do we need to make an uncountable number of choices? We can imagine doing a countable number of choices, but an uncountable, what does it really mean? If we would instead be asked to do it for the algebraic numbers we would be able to devise some formula to do it, but clearly we are stymied if we try it for the real numbers. If we consider the construction on the circle we get a subset V of the circle and a countable number of disjoint translates of that set V. If V is Lebesgue measurable, the assumption of measure zero or strictly positive measure lead both to a contradiction, and the most reasonable thing to do is to consider it unmeasurable[19]. This may or may not be thought counterintuitive or paradoxical. Then we can twist the argument a bit more, and subdivude a ball into a finite number of pieces and reassemble them to a ball twice that big (and by recursion arbitrarily big). What to make of it? It certainly has nothing to do with physics. Or on a more purely mathematical side, we can well-order any set. Can we really imagine an uncountable well-ordered set? [20]. But of course the purpose of mathematics is to make imagining the unimaginable possible.

Logic had until recently been part of the elementary instincts of a mathematicians as explained by the Pierces, now it was for the first time really challenged. How to salvage mathematics which seemed to be crumbling due to faulty foundations? There seemed to be no way of independently checking those monsters of the imagination. And logic was indeed revived trying to rise to the challenge. Even earlier it had received another new impetus, namely how to mechanize logical reasoning. This history goes back at least to the 17th century. Pascal made an attempt at a calculating machine, and so did Leibniz, who more interestingly thought of some rational and universal language in which people could calculate and sort out their differences. The real modern advance was that of Babbage who devised an algebra of reasoning, which lies at the foundation of modern computers, namely through the mechanical way simple logic could be simulated. The two strains were of course independent but would work in tandem. There was a metaphysical strain to logic, closely related to the moral we have referred to as earlier, as well as practical and pragmatic, and both became intertwined. The metaphysical strain was philosophical, having to do with analyzing the ways we think, and what thinking really entails. The practical strain was related to engineering, and concerned formalization. The notion of formalization has of course a philosophical aspect. Why do it? What does it mean? But to actually do it is honest work like anything else in life and mathematics. Thus logic was called upon to both morally support mathematics, as well as in practice become a part of it, applied mathematics in a sense.

The obvious thing to start was to axiomatize even simple and intuitive things such as the integers. This entails some formalization as well as some philosophical reflection. And as noted the ultimate purpose need not be to attain absolute truth (finding God?) but to make things more transparent and open to scrutiny. To seek is better than to find.

The axioms of Peano are well-known. They can be summarized somewhat informally by

A) 0 *is a natural number*

B) i) Every natural number n *has a unique successor* n'.

ii) If $n' = m'$ *then* $n = m$.

iii) There is no n *such that* $0 = n'$

C) If P *is a subset of the natural numbers, with the property that if* $n \in P$ *then* $n' \in P$ *and* $0 \in P$ *then* P *is the set of natural numbers.*

Given this one may define $m + n$ inductively as through $m + 0 = m$ and $(m + n') = (m + n)'$ and $m \times 0 = 0$ and $m \times n' = m \times n + m$ and easily prove the commutative and associative laws of addition and multiplication as well as those such as the distributive that connect the two, which would be very difficult otherwise[21]. For completeness sake one should also mention the order, as the natural numbers are naturally ordered, the ordering more fundamental than the arithmetic properties. It is intuitively clear what is meant by $m < n$, namely that n is obtained from m by successively using the successor operator.

But how to formulate this in a formal way, using only the language of Peano arithmetics? One solution is to say that $m < n$ if there is a number $c \neq 0$ such that $n = m + c$. But what does that really mean, if not the thing above? But we are less concerned about what things mean, than how you manipulate them. Just as with the now forgotten Marshal McLuhan and his 'the media is the message', the meaning is to be found in the manipulations.

Now the last statement is very different from the other two, which can easily be implemented in a mechanical way. Nowadays it is thought of as a second-order statement, involving the notion of an arbitrary set. Once again we see that by formulating something explicit we are in a better position to criticize and hence modify. Thus in the modern version of Peano we have a first order statement. In fact an induction schema, which consists of a countably infinite set of axioms. For each formula $\varphi(x, y_1, ..., y_k)$ in the language of Peano arithmetic, the first-order induction axiom for φ is the sentence

$\forall \bar{y}(\varphi(0, \bar{y}) \wedge \forall x(\varphi(x, \bar{y}) \Rightarrow \varphi(x', \bar{y})) \Rightarrow \forall x \varphi(x, \bar{y})) \forall \bar{y}(\varphi(0, \bar{y}) \wedge \forall x(\varphi(x, \bar{y}) \Rightarrow \varphi(x', \bar{y})) \Rightarrow \forall x \varphi(x, \bar{y}))$

We use the abbreviation $\bar{y}$ for $y_1, y_2, \ldots y_k$

Now every φ clearly represents some formula to define a set, and those formulas will be rather complicated. Thus we can pose a couple of naive comments.

First, something so simple as the integers are defined by something much more complicated[22].. Also what does $y_1, y_2 \ldots y_k$ refer to? A finite number of variables, and k to an arbitrary number. Thus we try to define numbers, by assuming an intuitive knowledge of them. As such, such an ambition is not to be deplored, on the contrary, but if taken literally, as is the point of formalizations, it is hard not to voice a certain concern about circularity. Apart from that it is clear that no one, who does not already have a firm understanding of numbers will understand what is going on.

Now to be less naive. The point of those axioms is not pedagogical. It is an attempt at formalization, to provide a language with which to speak about numbers and prove things about them. As we will see the metaphysical notion of truth is not dependent on proof, something is true or not independent of proofs, at least if we take a strong ontological view of mathematics to which we will return. Thus it should be thought of as an engineering feat not as an act of salvation. In fact the point of this cumbersome formulation is to avoid a certain self reference. The sets which are considered, are not 'all' subsets of the integers, only those which we will be able to formulate and define. In fact as there are only a countable number of formulas, and the number of subsets of the integers are uncountable, most will be missing. With the strong second order axiom all models of Peano are isomorphic, a fact already proved by Dedekind[23]. But with the weaker form, one will have many models, all containing the standard model as an initial segment. One speaks about non-standard models, and non-standard numbers, numbers in fact beyond infinity. The fact that you have models in one, not present in the other, shows that those two systems are fundamentally different. The first one is very categorical, relying on a general principle, there will only be one set of numbers. The second one is more modest, it only entails a limited ways of speaking about sets of natural numbers, limited in the pragmatic way of what can actually be said, with no resource to anything supernatural. To anticipate the discussion below, one may think of the meaning of a text to be external to the text, in fact being the 'soul' of its 'body'. While the formalistic approach is to focus on the text as a 'body', what it really entails by itself. As noted above, meaning is to be found in the way things operate. Formalism is a kind of materialism. Of course the drive to formalize, and the way to think about formalization, is not part of formalization proper.

Now as to some standard models of the axioms. We may define a set $\mathbb{N}$ of sets as follows.

A) $\emptyset \in \mathbb{N}$

B) Given $S \in \mathbb{N}$ the successor S' is given as $S \cup \{S\}$

Of course in a strictly logical way this presupposes an axiomatization of the sets to which we will return later. Thus we have $\emptyset, \{\emptyset\}, \{\emptyset\} \cup \{\{\emptyset\}\}, \{\emptyset\} \cup \{\{\emptyset\}\} \cup \{\{\emptyset\} \cup \{\{\emptyset\}\}\}, \ldots$ it certainly becomes unwieldy. Are there any points to it? One point is that we can state that $A < B$ iff $A \in B$. But to define the numbers as those particular objects seems rather autistic. Why not define S' as $\{S\}$ or if we are only concerned about marks on papers to start with the empty mark and given the mark S to defineS' as $S|$ getting the much more readily interpretable $, |, ||, |||, ||||, |||||, \ldots$. The discussion seems to concern more designation of numbers than numbers as such[24].

A different proposition was advanced by Frege, another one of the pioneers of mathematical logic, who basically suggested that a number should be identified with the set of all sets with the same cardinality. As we noted there are problems with such sweeping definitions of sets, and hence it does not pass muster from a formal point of view.

Now in order to build new foundations for the new kind of mathematics and reasoning, a variety of remedies were proposed. The mathematician Hilbert, whose grasp of the problems was in my opinion the firmest of them all, spoke about the paradise created by Cantor, and the desire not to allow oneself to be expelled from it. Other mathematicians, such as Poincaré had nothing but contempt for logic, and greeted the paradoxes and contradictions with some glee, noting that at least it started to become interesting. He also noted that the quandaries besetting the logicians were essentially no more complicated and profound than those expressed by the Old Greeks through the Liars paradox. And indeed the self-reference (or almost self-reference) that lies at the center of Cantor's diagonal trick, plays a pivotal role in all of modern logic. So in many ways the sarcasm of a Poincaré was not misplaced.

The traditional way of dealing with problems is to devise laws. Laws that regulate actions, and in this case thought. To forbid certain thoughts is not necessarily a bad thing, as I have repeatedly pointed out, there is nothing to challenge and hence stimulate the imagination as obstructions to be overcome.

The most radical approach was that of the so called intuitionists. They sought to do away will all non-constructive approaches to mathematics. Pure existence proofs were to be considered meta-physical, thus monstrosities such as a well-ordering of the reals would be banished. This was done by renouncing the excluded middle, thus making it impossible to use proof by contradiction. In other words if P is false does not necessarily mean that $\neg P$ is true. Formally $\neg(\neg\neg P = P)$[25]. If our method of ascertaining truth is by checking a finite number of cases, we can check that a given diophantine equation has a solution, but we can never check by a finite number of cases that a diophantine equation does not have a solution. Thus if it has we will eventually know, if not we will never know. The intuitionistic ambition is to do away with faith and the resort to supernatural methods, but of course the whole approach is based on some metaphysical assumptions. As the Britih historian and philosopher R.G.Collingwood quipped; 'Those who reject meta-physics take a meta-physical stand'.

> How can we doubt the fact that if we add one to an even number we get an odd number? But it is a statement about an infinite number of numbers and we cannot check them all? What about the 'proof' $2 \cdot m+1 = 2 \cdot m+1$? It looks as a tautology. Formally it is, but we need to interpret it. On the left hand side two things are going on. First the claim that an even number can be written under the form $2 \cdot m$. This we may take as a definition of 2 being divisible in the number, or that it is even. Then we add 1. If we interpret that the right hand side shows an odd number by definition, we really have not accomplished very much. We have indeed merely belabored a tautology. To give some real content to the assertion we have to show that there are two different types of numbers, even and odd, that any number is either even or odd, but can never be both. To show that a number cannot both be even and odd, we need to show that 1 is not even. $2 \cdot 0 = 0$ and $2 \cdot 1 + 2 = 1 + 1$ thus $2 \cdot 1 > 1$. Now assume that $2 \cdot m > 1$ this means that there is $c \neq 0$ such that $2 \cdot m = c + 1$ then $2 \cdot m' = 2 \cdot (m + 1) = 2 \cdot m + 2 = c + 2 = (c + 1) + 1$ thus $2 \cdot m' > 1$. Then we invoke the induction principle to the set consisting of 0 and $m : 2 \cdot m > 1$ and conclude that it is the whole thing, and thus there is no number m such that $2 \cdot m = 1$. Even when I have skipped a few steps, the whole argument seems rather involved to prove such a diddly fact, a mere indulgence in pedantry. But of course the point of the exercise is not to convince yourself of the truth of the proposition, anyway it is doubtful you would believe if you were not convinced before, but to make our reasoning more transparent, whre actually does infinity enter in the reasoning? This is of course clear, it is the induction axiom, which makes it possible to speak about properties of infinite set of numbers using a transcendental principle, which we as little can check one by one, as we can prove induction by induction (as David Hume remarked, much to his youthful dismay). Now, we are cheating in a sense, we are using the sloppy (second order) version of the induction axiom. What about using the first order scheme? Now it becomes a bit more challenging, we have to express the set using a first order formula. This takes some work and ingenuity, but is of course not insurmountable. Anyway it should be clear that the focus of attention has shifted from the result to the structure of the proof.

We are thinking of the proof as such as a mathematical object. To this we will return.

As can be understood intuitionism became an ideology, a kind of mathematical religion, and the Dutch mathematician Brouwer became its prophet[26]. Some great mathematicians, such as Hermann Weyl, were intrigued and joined the movement, but at least in the case of Weyl it does not seem as if it really effected their mathematics, the pursuit of which touched far deeper stratas of their personalities. For many mathematicians the strictures imposed by the intuitionists were felt as a constraining straight-jacket and thus too uncomfortable to wear. On the other hand, following the adage of obstructions stimulating the imagination, over the years there has been systematic attempts to prove as many of the classical theorems in analysis as possible using only constructive methods. In many ways this has been a stimulating exercise and the underlying idea being that theorems that consistently resist such efforts may after all be irrelevant. As an elementary example we can take the theorem of intermediate value of continuous functions. The proof given above essentially works in the constructive realm, one only needs to make some reinterpretations of the concepts involved. First we should no longer think in terms of exact real numbers, i.e. infinite sequences of decimals, or whatever is to your taste, but approximate. And to have a function means to have a formula for calculating it, not just any arbitrary set of numbers, because such functions only remains in the imagination beyond the imagination, i.e. they can never really be pin-pointed without wa(i)wing your hands. As a result any function will be continuous. This may be strange, what about the function $f(x) = 1$ if $x \geq 1$ and $f(x) = 0$ otherwise. A perfectly explictly given function, very easy to imagine and visualize. But the point being that we cannot locate exactly the value 1 if we think in terms of approximations. Thus the function is not really defined there. But if $x < 1$ then eventually there will be no unending sequence of 9's and if $x > 1$ no unending sequence of 0's. And if the number 1 is removed there will be no discontinuity. Now with this in mind, when we subdivide the given interval successively looking out for changes of signs, we need to be able to compute the functions to the appropriate accuracy, and we are less concerned with the existence of an exact value as lying in all the nested intervals, then to at each stage get approximations given by the size of the intervals. The computation of $\sqrt{2}$ done in a previous endnote is a perfect illustration of the principle. We are not concerned about the exact value of $\sqrt{2}$ involving an infinite number of decimals, (or dyadics as in the example), but only about the successive approximations, pursued to the extent which is appropriate to the situation. It is indeed a beautiful example of the ϵ, δ rigmarole which is about the potential infinity not the actual[27]. The fact that this process can be continued indefinitely is a meta-physical statement, transcending the approach, but of limited practical concern, except as to faith.

Pedantry aside, which incidentally is not the point, the intuitionistic approach does make us ask some questions we may not have thought about otherwise, and as such it provides, for all its possible perversity, a stimulation.

Inspired by the example above, what sense does a general real number have? To put it down we need to make an infinite number of decision as to what to pick. One may think of it as the outcome of an idealized toss of a coin pursued indefinitely, or by invoking Zeno, as done an infinite number of times in a finite time. Can we dispense with those and only think of real numbers as sequences for which there is some formula? The example of the $\sqrt{2}$ is an excellent example. The procedure described above (in an endnote) is deterministic and allows the writing down of any number of dyadic digits. Yet it is subtle enough to give the impression of chance, as the procedure gives no general picture of any patters of digits (as in repeated decimals). Now if we restrict our notion of real numbers to this, as the other kinds will never enter into our concerns (it is like throwing away a large number of numbers in the kind of printed catalogues of the past known as telephone directories), we get another model, actually a countable model of the reals. But how could a countable model be a model of something uncountable? This may be seen as a paradox. It is not really a paradox in the end, but when such it will challenge our notions of cardinality, and what it really means. The point is that if we do enumerate them all, this enumeration cannot be given by a formula of sorts, because if it did, we would indeed by the standard diagonal trick exhibit a missing number. Thus this model (introduced by Skolem and Löwenhielm, and then vastly generalized to any first-order system) is countable, but does not 'know it' from within. Thus if you want we may think of it as a parable. Only God knows the countability, being able to use second-order language, while humans, constrained to first-order, see it as uncountable. As have been noted before ad no doubt will be noted again, the medieval scholastics and modern mathematical logicians, share many things in common!

Now to return to the moral basis of mathematics, from having digressed on the intuitistic approach, which should be thought of as epistemological rather than otological, thus occupied with what can be known, rather than with what is. Logic was thought to be the solution. After all mathematics is logical, and should be thought of as applied logic. Frege started this train of thought and it was brought to conclusion by Russell and his older mentor Whitehead. The result, known as 'Principia Mathematica' (with an obvious wink to Newton's *Magnus Opus*) turned out to be a *cul de sac*, at least from the view point of a mathematician. In fact a giant monument stranded on land (like the first attempt of Robinson Crusoe to fashion a seaworthy vessel to take him away from his island) not to be carried to the sea. It took a great toll on its authors, who never returned to logic on the cutting edge[28]. Russell was rather defensive about the work, on one hand claiming that only six people had read it through (all Poles incidentally), which is hardly surprising given the highly technical presentation[29], on the other hand that it contained some gems of transfinite arithmetic also of purely mathematical interest. May be, but if so a rather meagre outcome of so much effort and ink. But ruins have their usefulness as quarries for future buildings, and some ideas of the work made a more lasting impression. Russell is known for his theory of types, designed to prevent the formulation of his eponymous paradox. Basically it entails a stratification of sets, a set only containing sets as members of lower strata. One unfortunate technical consequence was that there were different empty sets, one for each type![30].

Of all the people involved in the foundations of mathematics at the turn of the century, one name stands out, namely that of Hilbert, whom I think had the firmest grasp of the problems. von Neumann may have been more brilliant, but brilliancy is to large extent a matter of show, and I dare say that Hilbert was more solid. He has been described as a formalist, and although the designation certainly can be justified, it is nevertheless misleading. It is true that in his description of axiomatics he very much stressed the formal nature of the objects, that they by themselves did not mean anything, they only acquired an instrumental meaning through the ways they interacted. Russell took this notion *ad notam* stating that mathematics was a formal meaningless game, that mathematics was just a string of tautologies, and refered to nothing, and when we speak about mathematics we do not know of what we speak. This high-minded philosophical attitude, may easily be mistaken for brilliancy, and that surely was the intention (among others), but it was not one shared by Hilbert, and one more natural to one only with a relatively superficial relation to mathematics. There is little to indicate that Russell ever was a serious student of mathematics (even less so as to Wittgenstein) unlike most people working in mathematical logic at the time. To Hilbert formalization of mathematics had one specific purpose, namely that of making proofs and mathematical reasoning as far as possible objects of mathematical study, with one particular application in mind, namely that of proving consistencies of central axiomatic systems. Thus rather than thinking of mathematics as applied logic, Hilbert turned this upside down, thinking of logic as applied mathematics, ideas to some extent implicit in Pierce[31]. With Hilbert came the distinction of the formal language in which a theory is couched, its intrinsic one so to speak, and the meta-language with which we speak about the system extrinsically. This is a distinction we have already encountered in the countable Löwenhielm-Skolem model of the reals above. In a way this is a very Cartesian approach, in which we make a distinction between the body of the formal system, which is mere material, and the meaning with which we imbue it, which we may in fancy metaphors refer to as the 'soul'. A system of axioms, as a formal system, a game if you want; which may refer to nothing, is as such a thing in the real world, more or less physical. Of such a system you may ask real mathematical questions. In fact the most interesting one, is it consistent? Thus the question of consistency is not an idle question having a different set of answers depending on your mood, but a very palpable one with a definite answer, may it be permanently evasive.

Although Gödel can hardly be thought of as a student of Hilbert, he nevertheless must have taken Hilbert's approach seriously understanding what it was all about. What Gödel really did was to try and embed the meta-language of a system rich enough to contain the integers, into the system itself. This means translating statements, say about the integers as a system, into arithmetical statements. The embedding was not perfect, but strong enough to allow some startling consequences to be drawn. Namely that there are true statements about the integers that cannot be proved, and in particular the consistency of integers as a system, being translated into a statement about the integers being one. And of course more generally for any system containing them.

Now this made a stir in the mathematical community being prepared for it. When presented at a meeting, von Neumann immediately saw the implications (in consistency, which had not been immediately appreciated by Gödel), and Hilbert rather than being devastated by his program being dashed was delighted. This is how true mathematicians react, not imprisoned by ideological constraints, but excited by the spirit of mathematics triumphing.

The proof of Gödel's theorem is not particularly difficult as far as mathematical proofs are concerned, for one thing it is not technically involved, and the concepts involved are natural and require no extended sojourn of familiarity. One great idea is the so called Gödel numbering, which is trivial to implement, and which forms the point of departure. The real difficulty of the proof is to keep in mind simultaneously the formal meaning of a sentence and its real meaning and the 'true' relation between the two. Thus it is one of conceptual difficulty (which might have been too taxing for Russell and Witgenstein). It also depends on our ability, or maybe rather inability not to, to consider an infinite sequence of cases one by one, which cannot be formalized into the proof but instead provides the irresistible conviction of its inevitable conclusions.

Much hype concerns the proof and what it really means. This is due to the fact that the context within it sits is relatively accessible and hence has attracted a lot of comments from busy-bodies. One such interpretation is that mathematics in a sense transcends logic, that there are truths beyond logical reach (*There are more things in Heaven and Earth, Horatio, than are dreamt of in your philosophy* Hamlet I:5), and the infinite creativity of mathematics. I certainly personally agree that mathematics transcends logic, but that is nothing that you can derive from Gödel's, the way you derive a corollary from a theorem. You can possibly conclude it from the spirit in which it is proved. Penrose uses it to debunk the strong claims of the Artificial Intelligence that the human mind can be simulated on a computer, i.e. that thinking cannot be reduced to an algorithm. Much as I am in sympathy with his quest, the argument does not really carry more weight than the man in the street can produce by referring to his sense of humor. As noted our propensity to think in terms of infinity is central to the persuasive power of the proof. True this cannot easily be formalized, but maybe the human mind is limited in its ability to formalize (which of course amounts to the same thing), and to be able to specifically determine the boundaries of human thought, seems to be a way of actually going beyond them. The point is that Gödel's proof is about formal systems, not more nor less, and that few things in life is amenable to formalization. Nevertheless Gödel's proof is fertile ground for metaphors, and that is perfectly legitimate.

The crucial construction in Gödel's proof is a variant of the Cantor diagonal trick. In fact Gödel claims to have been inspired by the Richard paradox proposed in 1905 by the French mathematician Jules Richard (1862-1956). There are variants of it but let us formulate it as follows[32]: **Let N be the least inte-**

ger that cannot be defined by less than twenty words in the English language. This is a sentence of nineteen words that unambiguously defines a natural number. A pedant may prefer **Let N be the least integer that cannot be defined by less than one 106 characters in an English sentence.** But the sentence has only 105 characters. As there are only a finite number of sentences with a fixed number of characters, only a finite number of integers may be defined by such sentences, so the set referred to is non-empty. Now the problem is what number should we assign to a sentence like 'Hamlet and Horatio are characters in a play by Shakespeare'? Maybe we should only assign numbers to really relevant sentences? Sentences such as 'the sum of all numbers less than a million', 'the total number of scenes in the play Hamlet', 'the biggest prime factor in thirtythousandfivehundredandtwenty'. But what about sentences like 'the smallest odd perfect number' when we do not even know whether there are such numbers. Thus the sentence may or may not refer to a number. We are obviously concerned about what is, not what we may happen to know. One way of getting around it is that we simply make a list to which we will assign to every sentence a number by fiat. Now given that list the sentence above actually will define a number, but there will be a conflict. On one hand we have the formal assignment, and on the other hand, the real assignment. The first refers to the internal language of the system, the latter the external meta language, and those should be kept apart. Note that there is no infinity involved, as we are only looking at sentences with a bound on their number of characters. We can interpret this in a number of different ways. If the formal assignment would be done in a true way, thus throwing away irrelevant sentences, and having infinite wisdom, thus knowing the smallest odd perfect number if it would happen to exist, the making of the list would in a sense be a 'new' thing in the world, which only now can be taken into account, and thus give new references. The list refers to itself, or alternatively, every time we make up a list, we change the basic assumptions, and have to make up a new list. Languages are notoriously vague and ambiguous and always needed to be interpreted by humans.

> Could we not instead of sentences in a human language consider sentences in some formal language, or maybe even computer programs? There are three types of computer programs. Those that never get off ground (they are faulty), and those once they take off never land (i.e. those that never halts and hence never produce an output) and the final ones which take off and eventually halt and then produce an output, which we can always interpret as a number. In this way we would set up a purely mechanical system. The process itself is mechanical, and thus could it not be mechanically generated by a computer program? The computer program would emulate all the programs involved, say by generating their codes and then run them. Then it would produce a finite set of numbers, and look for the smallest one of those missing. Now if this super program would be of a given length, and generate all the programs of that length or less, we would have a formal, mechanical version of the Richard paradox. [33]. The

problem is that in order for that to work, we need not only to reject faulty programs, but also to identify those which never halt (an this program which recreates itself in an unending nesting continually updating itself) will never do so. In this way, as Turing showed, you can conclude that there is no such program that can identify a non-terminating one from the outset.

Now the consistency of a set of axioms can in principle be settled mechanically by a program that systematically goes through all possible logical chains of the system. If it stumbles upon a contradiction it outputs 0 for faulty. If it never outputs anything it is consistent, but we will never know, the program fails to output anything. On the other hand if it is inconsistent, it will eventually stumble upon it, and we will have a finite proof of the fact. Likewise more directly arithmetical propositions, such as that a certain diophantine equation does not have any solutions, will if false have a solution, and that very fact would provide a finite proof (through exhibition) of its falsity, but if it would be true, we would not necessarily know, it would be undecidable. Thus undecidable questions are necessarily true! If we can prove their undecidability, we have also proved their truth, but then they are not undecidable any longer! Anyway the proof of this fact appears to be on a higher level than those proofs we normally consider. It seems to be a proof on the level of the proof structure of proofs. We could imagine that we find some fixed point theorem working on putative proofs, and the fixed point would be the proof, but we would have no idea how to pinpoint it. In other words this is not really mathematics but metamathematics.

I have not given any attempt at giving a proof of Gödel's theorem, such are, however, readily available, but I hope giving some ideas of what is involved and the taste of the reasoning. Now what impact have Gödel had on mathematics? In fact not very much. It has, however, had a tremendous impact on logic, infusing it with mathematics, in fact indicating how logical questions can be turned to mathematical, changing the subject to a species of 'applied mathematics'. The original unprovable proposition in the proof was a sentence that proved its unprovability. If such a sentence would be false it would be provable and hence true (assuming consistency, otherwise everything is provable, including the consistency of the system), thus it needs to be true (excluded middle). This if of course analogous to the statement 'There are no truths' which cannot be true, hence false, hence there are truths! Later less contrived undecidable mathematical statements have been produced. This is mathematics of course, but mathematics more interesting to the logician whose subject is mathematical reasoning, than to mathematicians, who generally are not, as Pierce pointed out, especially concerned with mathematical reasoning *per se*. Mathematics as a subject grows organically, and it has been one of the deepest mysteries that mathematical problems are usually solvable, that so many true statements are actually provable. The unprovable belong to the more or less lunatic fringe. This is why we expect natural problems, such as the Riemann hypothesis to eventually have a conclusion. This lies behind Hilbert's famous dictum *wir wollen wissen, wir werden wissen*. Thus mathematics more or less goes on along its own way without paying too much attention to logicians. This does not of course mean that they are uninterested, just as a mathematician may be interested in literature, history and contemporary political and economic issues, without necessarily integrating them in his mathematics, he may also have an intermittent interest in mathematical logic, feeling that it is necessary for his wider culture.

Now mathematical logic has left two marks on mathematics. One is the adoption of set theory as the language of mathematics. Nota bene, mathematics is not a language, as little as physics and chemistry is a language, but it has a language and that is the language of sets. There are mathematical questions you can ask about its language, but most mathematicians are not really concerned about it, especially not higher cardinalities. As the French mathematician Dieudonné put it, all what the analyst needs to know is the uncountability of the reals. The language of sets was used by Bourbaki, whose members conceived mathematics in the form of structure,which could be expressed as nested sets of subsets. The style itself may also be said to be influenced by the formal logical approach, and pride itself on its abolishment of pictures. This has come in for much criticism, much of it justified, some of it exaggerated and misplaced.

If set theory is ostensibly the basis for mathematics one expects a set of axioms for it. But while geometry more or less has a unique structure (let us disregard for the sake of argument the non-Euclidean version) which we would like to see as a description of an underlying physical reality, the situation with set theory is very different. There are many different axiomatizations, and many different universes of sets, and which one is the canonical one seems to be more a matter of convention than anything else. At first sight this seems to go against an objective ontology of mathematics, if mathematics is ostensibly founded on conventions, and hence subject to the whims of human imagination. To this question we will return.

The second mark that has been left on mathematics is computers, which allow a reification of mathematics. The spirit turned into flesh. Although in principle you cannot prove things by computers[34] you can test things. And to many mathematicians numerical corroboration may be more persuasive than long formal proofs, mathematicians are but human after all. Initially it was thought by the general public that computers would replace mathematicians. This reflected a deep ignorance of what mathematics is, and to a lesser extent, what computers are capable of. In the early years of the computers there were exaggerated claims as to what they would be able to do. The science fiction literature at the time abounded in stories of computers taking over, becoming more intelligent than humans. Ironically that was at a time when the actual performance of computers were far below what is possible today. Since then there has been roughly an exponential rate in the memory capacity of computers, their rate of conducting operations and a concomitant exponential decay in prices (known as Moore's law). This has made computers available as gadgets to the masses, who in the process has lost some of their erstwhile respect. Familiarity breeds contempt as is well-known. However, the familiarity with computers is very superficial for most people and is mainly limited to interaction with various commercial interfaces, and thus rather than sharpening the wits of their uses, tend to dull them. The heart of computers lie in programming, and the potential of soft-ware development has taken on a new level when speed and memory capacities have been vastly improved. Is it possible to design software which is conscious or at least intelligent, which has a will of its own? And if so if humans are able to design intelligence more intelligent than itself (whatever that means) would not that intelligence have the same ability, namely that of transcending itself? Would we not than have an exponential increase in intelligence, a veritable cognitive explosion? No trees are supposed to grow to heaven, and can this exponential increase go on unchecked? What is intelligence anyway? Does the ability to manufacture something more intelligent than yourself necessarily mean that you would do it? Maybe the process would stop? Or perhaps it would become impossible to continue, just as you can easily double the thickness of a paper by folding it, but this only works initially, you cannot reach the Moon by folding a paper say fifty times. The speculations above have something of the spirit of transfinite cardinals. A set creates its own powerset, more powerful than itself, which in its turn..

Admittedly there has been advances in artificial intelligence. Meaning that machines can do things that traditionally was considered the privilege of humans. A useful such is 'face recognition'. What has attracted most attention though are 'useless' feats as chess or go playing machines able to beat the best of humans. The sky seems indeed to be the limit. Now the triumph of artificial intelligence is more due to the ingenuity of the programmers than to intrinsic qualities of the machines *per se*. The idea of a general intelligence is one which would need no intervention by humans, no initial conditions of any specific kind. Instead the machines are meant to interact with the environment, learn and improve themselves by some boot-strapping process. The principle seems simple and natural in the abstract, just as natural selection, and has not natural selection during a geologically brief time produced human intelligence? But in practice to get anything off the ground quickly (geological time scales are not options) one need efficient search procedures to navigate a combinatorially vast universe of *a priori* possibilities. Natural evolution works blindly by selection, and the stricter you make the selection criteria, in the interest of speed, the more likely that you overlook potentially promising avenues.

What does this have to do with mathematics? On one hand it constitutes in particular an inquiry into what extent mathematical thinking can be mechanized. Traditionally much of mathematical research was done by computations. Straightforward computations which can easily be mechanized, which in effect already were (recall that traditional computations were made not by paper and pen but by devices such as abacuses, making the written representations of numbers irrelevant at least for computational purposes). There were early attempts to make calculating machines, Pascal and Leibniz are well-known examples. The lack of success was due not to the primitive state of their ideas but that of contemporary technology; there was nothing wrong with their visions. Algebraic manipulations are also rather mechanical, but things are more subtle here. While there are standard forms of the result of a numerical calculation[35] the end point of an algebraic formula depends on what uses it should be put. If you want to show that a given polynomial always is positive (for real values) the form $(x+1)^2+1$ is clearly more useful than the standard x^2+2x+2. Now logical reasoning in general also can be indulged in mechanically, this is what happens when you read a proof line by line. You have a feeling of having a local understanding but do not really 'get it'. It is like taking in a picture pixel by pixel. You understand each pixel, and its immediate relation to nearby pixels in the linear presentation in terms of which is darker and which is lighter, but the picture itself does not 'pop' up. It is notable that the metaphor for understanding is 'seeing'. as in 'Now I see what you mean'. Leibniz recognized this possibility. He upheld the computation as the examplary form of objectivity, envisioning a perfect language through which any two people having a difference of opinion could settle their differences by a calculation. Initimately related to this is Leibniz vision of this world being the best of all possible worlds, an idea ridiculed by Voltaire. This is closely related to the vulgar (i.e. simplified) conception of evolution[36].

Basically this is a question of materialism *par excellence*. Computers turn the spirit into flesh, which in the Bible is thought of as a miracle. But to do the opposite, i.e. to turn the flesh into spirit is considered a miracle of miracles. Will this happen? Will we produce artificial intelligence, in particular mathematical? The formalization of mathematics is, as noted above, an attempt to turn mathematics into body. To make it nothing but, as the classical Greek philosophers held, a combination of particles. Much of the classical approach to a symbolic language for mathematics and logic is inspired by that old Greek approach. We talk about atomic sentences, which are combined into larger entities, which in their turn combine, and so on. Mathematics lends itself particularly well to formalizations, can we then formalize mathematical reasoning? Just as the chess computers were the departing point for artificial intelligence, mathematical reasoning would be the logical next step, after all mathematics is just a game, if more intricate than chess and go. But is mathematics just a game? The axiomatic approach seems to indicate this, as this is a formal in the sence that the concepts are neither true nor false, just as the rules of a game, just conventions to be played with. To many mathematicians the notion of a game is a rather frivolous one, it takes away from mathematics its seriousness and its connection to reality. This leads us to the concluding question. In what sense does mathematics exist? To that we will return in the next section.

By first we may after this lengthy digression return to the initial theme - the foundations of mathematics. As far as Set theory constitutes the language of mathematics, its axioms should in a sense provide its foundations. At least it would give a short answer to the problem of the section. So here they are as presented by Zermelo and later modified by Frankel.

1. **Axiom of extensionality**

Two sets are equal (are the same set) if they have the same elements.
$\forall x \forall y [\forall z (z \in x \Leftrightarrow z \in y) \Rightarrow x = y].\forall x \forall y [\forall z (z \in x \Leftrightarrow z \in y) \Rightarrow x = y].$

2. **Axiom of regularity**

Every non-empty set x contains a member y such that x and y are disjoint sets.

$\forall x[\exists a(a \in x) \Rightarrow \exists y(y \in x \wedge \neg \exists z(z \in y \wedge z \in x))].\forall x[\exists a(a \in x) \Rightarrow \exists y(y \in x \wedge \neg \exists z(z \in y \wedge z \in x))].$

3.**Axiom schema of specification**

Given any set A, there is a set B such that, given any set x, x is a member of B if and only if x is a member of A and φ holds for x.

$\forall w_1, \ldots, w_n \, \forall A \, \exists B \, \forall x \, (x \in B \Leftrightarrow [x \in A \wedge \varphi(x, w_1, \ldots, w_n, A)])$ or in words:

4.**Axiom of pairing**

If x and y are sets, then there exists a set which contains x and y as elements.

$\forall x \forall y \exists z(x \in z \wedge y \in z).\forall x \forall y \exists z(x \in z \wedge y \in z).$

5. **Axiom of union**

$\forall \mathcal{F} \, \exists A \, \forall Y \, \forall x[(x \in Y \wedge Y \in \mathcal{F}) \Rightarrow x \in A].\forall \mathcal{F} \, \exists A \, \forall Y \, \forall x[(x \in Y \wedge Y \in \mathcal{F}) \Rightarrow x \in A]$

6.**Axiom schema of replacement**

The axiom schema of replacement asserts that the image of a set under any definable function will also fall inside a set.

Formally, let ϕ be any formula in the language of ZFC whose free variables are among

$x, y, A, w_1, \ldots, w_n x, y, A, w_1, \ldots, w_n$, so that in particular B is not free in ϕ . Then:

$\forall A \forall w_1 \forall w_2 \ldots \forall w_n \big[\forall x(x \in A \Rightarrow \exists! y \, \phi) \Rightarrow \exists B \; \forall x\big(x \in A \Rightarrow \exists y(y \in B \wedge \phi)\big)\big].$

7.**Axiom of infinity**

Let $S(w)$ abbreviate $w \cup \{w\}$, where w is some set.

$\exists X \, [\emptyset \in X \wedge \forall y(y \in X \Rightarrow S(y) \in X)] \, .$

8.**Axiom of power set**

By definition a set z is a subset of a set x if and only if every element of z is also an element of x:

$(z \subseteq x) \Leftrightarrow (\forall q(q \in z \Rightarrow q \in x)).(z \subseteq x) \Leftrightarrow (\forall q(q \in z \Rightarrow q \in x)).$

The Axiom of Power Set states that for any set x, there is a set y that contains every subset of x:

$\forall x \exists y \forall z[z \subseteq x \Rightarrow z \in y].$

Comments

First we note that the axioms are phrased in a very formal way, which is congenial to logicians doing mathematics, but is seldom used by mathematicians. One should once again emphasize the distinction between formulas used by mathematicians and formal language employed by logicians. The latter is used for precise description, while the former are used dynamically producing more formulas.

Secondly to a mathematician there is a touch of pedantry to the whole set-up, but from the point of view of the logician, that is the point.

The first is as noted a formal way of saying that sets are made up of elements, and it is those elements which determine the sets. This formulation may on the other hand be thought of as naive by the logicians. To the mathematicians sets are postulated and hence their properties are not self-evident and prior to man, but part of their definitions so to speak. On the other hand we should not forget the purely formal aspect to axiomatics, the sign $\in$ does not necessarily mean anything it is just an empty (i.e. *a priori* meaningless relation between two objects x, y potentially open to a host of different interpretations i.e. models. Let us consider as the objects the natural numbers in a more or less naive sense, meaning not based on set theory, and for which we have a clear notion of what $x < y$ means. Why not interpret $\in$ as $<$ and let us see how far it gets us. So far it satisfies the first axiom.

The second means in particular that we rule out anomalies such as $x \in x$. Here we have a strong condition already ruling out many constructions such as the set of all sets which from a naive point of view are perfectly reasonable objects of thought, not to say resulting from perfectly natural sentences. Is not everything we can express a legitimate object of thought? The axiom is in no way self-evident in the sense of being part of a deep seated intuition, on the contrary it is more of a convention, rather arbitrary as most conventions, and somehow a desperate attempt to prevent certain avenues of thought. Why this one, why not some other one, which may serve the same purpose? The first axiom seems redundant, the second seems obstructive, on the other hand we may look upon as a pill we have to take to prevent us from getting sick, so why not accept it on the authority of the medical establishment? Incidentally our primitive model satisfies the axiom, note that 0 is the empty set ($\emptyset$) and that every non-empty set contains the empty set.

As to the third axiom, it is properly speaking an axiom scheme, meaning an infinite number of axioms, one for each formula φ. As a naive mathematician one may wonder why not quantify over all formulas φ? Now pedantry comes in again. While a naive mathematician may think of sets existing by themselves without necessarily corresponding to some explicitly formulated condition (this is why we can think of an uncountable number of sets, while only getting our hands on a few), in a technical logical sense we can only refer to sets which are explicitly related to some property[37]. And properties in this sense confined to restricted languages, typical first order. Recall that the same phenomenon occurs for the strict formulation of the Peano axioms, which will have many non-standard models, as it does not allow us to speak about arbitrary sets, such acts requires a second order language, which would not be appropriate in giving axioms for sets. Thus to quantify over all formulas cannot be formulated in the restricted language, thus this intuitive reference to an infinite set of axioms, which really is metaphysical. Whether this is cheating or not is an interesting question to ponder. Finally we should note that we are not allowed to define sets by properties in general, we must first specify a given 'universe' so to speak, before we are allowed. This prevents us from formulating the Russell paradox, a toxic catastrophe, which this rigmarole is all about, and which created modern mathematical logic as a necessary discipline. Now, the reader may observe as there are only a countable number of formulas, there can only be a countable number of sets. Yes, externally, but not internally when we place ourselves inside the game, just as in the special case of definable reals above.

Now the rest of the axioms are needed to construct sets (note that the power set as such is not given the power set axiom, only a set you can use to define the power set as a subset of and then proceed by the axiom of specification, and similarly with the others), to enlarge your universe so to speak, and the only one in need of comment would be the axiom scheme of replacement. Now this is puzzling to a mathematician, after all is not a function defined as a certain relation between two sets, one given as the domain, the other as the target, so the image would automatically be a set. On the other hand the image may be a subset which is not covered by the axiom of specification, but nevertheless a natural object. We see that we are reduced to quibbling about what and cannot be expressed in first-order languages. Once again technicalities enters.

Now most mathematicians do not know of the axioms of Set theory (of which there are many suggestions) and to be honest could not care less. They play no part at all in their thinking, except of course in the intuitive sense as they form products and make subsets and unions and consider power sets to their hearts content, without worrying. So does it have any interest beyond that of curiosity and to keep the mathematical logicians busy? The cynical answer is no, in view of the fact that consistency will not be proved in any interesting case; on the other hand formalization of mathematics does have some interest, although one should not underestimate the difficulties of doing so completely, as there is the possibility of computer checked proofs[38].

Now the axioms above capture many of our naive and intuitive ideas about sets. Their purpose is to tame our imagination by proscribing what is meant by a description, such as in the axioms of specification and replacement and thus have it shorn of inappropriate excesses. But there are some natural questions as to whether there are a cardinality between countable infinity and that of the continuum (i.e. of the real numbers). This is known as the continuum hypothesis and was formulated already by Cantor. From a formal point of view the question cannot be settled, as Cohen famously showed in the 60's you can both assume it as an axiom or not, the result will be consistent (provided the original axioms are). In fact he provided models in which it was either true and false[39]. Our intuitive feeling is that 'all' subsets of the reals exist, and if so they all have cardinalities, so either nothing new beyond that of the finite, countable or the equipotence of the reals occurs, or there does. Both cannot possibly be true? Now in a formal sense, the fact that two sets have different cardinalities, does not really mean that one set has more elements than the other, only that there is a paucity of subsets, so no bijection between the sets can be found. It all has to do with the power of our languages with which we describe or equivalently construct sets.

More seriously though , the logicians love high cardinalities, and their existences are brought about by fiat, simply by adding axioms. One motivation for such a game is to 'prove' the intuitive consistencies of logical systems, ensured by a reference to a 'higher God' so to speak. Thus the above-mentioned analogy with Medieval scholastics becomes painfully obvious. Thus logic has both encouraged extreme generalizations, by reducing mathematics to a game on which it can act; as well as brought about caution through the need of paying closer attention to the languages employed. Neither belongs to the heart of mathematics, but as noted above, mathematics allows us to transcend our cognitive imaginative abilities, and why not also our imaginative conception of mathematics itself?

The Ontology of Mathematics

Geometry is literally the measurement of land. The palpable existence of land, or more generally space itself, is only doubted by idealistic philosophers when in a professional mode of thought and speech. The axioms of Euclid were an attempt to catch the essence of space in a few simple principles as to make reasoning about it independent of physical measurements. The results were ultimately grounded in physical reality and apprehended by visual intuition. The question of ultimate existence did not really enter. However, in Euclidean geometry there are idealizations, such as completely straight lines without any thickness whatsoever, and it deals with triangles, and other geometrical objects, which do not really occur in nature. You cannot represent those basic objects physically, but nevertheless they are thought to exist. But so if in what sense? One obvious answer is in our mind, thus as elements of our imagination. But usually we think of our creatures of the imaginations to be phantoms subjected to our whims. We think of things like wishful daydreams, possible scenarios, with only a tenuous connection with fact. But in the reasoning induced on us by Euclid, we are constrained. We still may indulge in wishful thinking and hypothetical scenarios, after all a proof by contradiction is the creation of an imaginary world whose only object is to be annihilated into non-existence. The objects which we entertain in our minds are restricted in a sense that seems even more unforgiving than that imposed by the real material world. The deeper aspects of our thinking seem independent of us, not subjected to our wills. As the psychologist William James, no doubt inspired by Schopenhauer, talks about the 'Will to Believe', but we cannot will ourselves to believe arguments which are not correct. We might as well try to will ourselves to believe in God. It will not work. True, we might will ourselves into willing us to be deceived, but that is different[40]. Plato, had as we all know, a solution to the quandary. He spoke about a real world beyond that of the senses. A world of ideas, or better forms, separate from the confusing world of the senses. The perfect lines and the perfect triangles dwell in a world of forms, and in the real sensual world, only degenerate copies of it are to be found, like the crude imitations of lines drawn in the sand. Now Plato realized that mathematics more than anything else lent itself to this philosophy, and Plato, as noted before, held mathematics in very high regard, and although not a notable mathematician on his own, very well plays the role of a patron saint for mathematicians. The idea of a chair and a table, even if clear in everyday situations, ends to become silly when carried to logical conclusions. Yet, ordinarily we do recognize a chair and a table when we see it, even if they come in many very different varieties, most of which we have never encountered before, and they can be quite deformed (like letters). Somehow we can 'see' the 'essence', just as we can see the 'essence' of a letter (this is why deformed letters are presented to us when a computer wants to make sure we are not robots!) or actually of an artist, whose style we may sense even if we have never seen the picture before. This philosophical approach is referred to Platonism, and is usually denigrated by philosophers and scientists, as being primitive and naive, and at best serving a didactic value to the beginning philosopher. The reason being that everyday examples like tables and chairs do not hold up to sustained scrutiny, and in biology the notion of species becomes fluid when seen from a historical evolutionary perspective. However, Platonism has in fact imbued science once it transcended a mere descriptive function and formulated general principles, in the physical sciences, codified as laws, which could be seen as trying to catch an underlying reality though the confusion of sensual impressions. The purpose and prestige of science is to go beyond mere surface appearances. The problem with much of criticism of Platonism is to take it too literally, to consider its historical manifestation, instead of the platonistic form, so to speak, of Platonism. Plato was but a human, and any attempt to clothe his vision in words, is bound to suffer the same fate, as a line drawn in sand, or chalked on a board. Mathematics is, as Plato recognized, eminently suitable to be seen in platonistic terms, as mathematics seems to be guided by some transcendent spirit, and all our attempts to present it, are bound not to be perfect. Mathematics is also a human activity, and as such different from the elusive subject of mathematics itself. Doing mathematics has the same relation to mathematics as the painting of an image has the same relation to what is being painted. Presentations of mathematics are faulted for being not rigorous, if mathematical objects and relations (there is of course no real distinction between them, as it would be in the physical world, where only the former would have palpable existence) would not exist, what meaning would this objection really have? The very criticism is imbued with an assumption of an underlying reality.

Now in what sense do numbers exist? Numbers arise in the context of counting objects which are permanent (at least as long as the process of counting lasts) and distinguished from each other. If there is no sense of individuality, or if the objects are constantly moving, merging and splitting up, counting becomes not only a frustrating but pointless quest. Thus numbers are in that sense intimately associated with cardinalities of sets which can be represented physically. This was at least the attitude of the Victorian philosopher and social debater J.S. Mills when he dabbled in mathematics and logic, and also the attitude taken by non-platonists. This ties up with the representation of numbers, as discussed above. When we have the primitive notation with a stub for each number, we are actually producing by marks on the paper, literal (and physical) representations of sets representing cardinalities. The interesting thing is that if we want to perform calculations using such marks, by its very unwieldiness, it will be prone to mistakes. Doing things in a more sophisticated mathematical way, will be much more successful. But of course the very notion of successful implies that there is a right answer, that the answer has some meaning directly related to the physical world. Thus connecting numbers to physical objects, we more or less (the less referring to the process of connecting) make them 'real'. In the language of logic, we make physical models for them. Now modern computers present good examples to make a distinction between small numbers with physical connections, and large ones, with none. If you program you often produce arrays. An array is an ordered (in fact numbered) sequence of objects (which typically are numbers). The length of an arrow is limited for physical reasons, because you must make physical space for its members by assigning space in the memory, which is a physical entity. A typical small number is a million, and I guess a billion can be used, and slightly bigger numbers, but there are limits to the upper size of a computer after all, at least in present technology, we could hardly make a computer bigger than the Earth, as well as limits to the lower size of a memory slot, because after all atoms are finite objects, usurping minimal space. But now if we have an array, we can make it represent numbers in the obvious way (just think of the binary representation). Those represented numbers can be huge, and as such there will be no physical sets whose cardinalities they can be seen to represent. In particular, although it is trival to design a program that exhibits them one by one in ascending order, the life time of the universe will not be enough to represent them all. But do they exist? They certainly exist in a physical way, as they have physical representations. We can add and multiply them easily, those are trivial programming exercises. We can work with them explicitly and every operation will exist, by virtue of having been done. What we do with them can actually be justified in a palpable physical way. Every such representation exists potentially, but do they all exist actually? We cannot explicitly pinpoint any representation that does not exist, because doing so would automatically make it exist!

> The reader may be reminded of a short story by J. Borges by title 'The library of Babel'. Any book can in a variety of ways be encoded as a number. In fact by an array of 0's and 1's if you prefer. Thus an array, given physically by a sequence of states at a few million memory addresses, actual physical spaces, may on one hand represent a number, and in a slightly less canonical way a book, i.e. a sequence of letters and signs[41]. Do all of those books exist? Borges imagines they they do in a giant library. Now there would not be space in the universe for such a monstrosity, never mind the gravitational pull it would exert on itself collapsing into a black hole.[42]. Now such a set cannot exist physically, but we can easily imagine it in our minds, and ask all kinds of questions about it. Now there is a difference between having a book unread on a shelf, collecting dust, working as an insulator, and brightening up your room with its colorful cover, and actually reading it. Among those books, the book you are now reading will appear countless number of times, in different translations, including Old Norse and the language spoke by the elves in a story by Tolkien, and countless variations of those. There would also be many interesting books, perhaps one which gives you the proof of the Riemann Hypothesis (with one or two subtle faults, interspersed by some bawdy verses in an unwritten play by Shakespeare). Of course a book as a book, as a conveyor of ideas, rather than just a formal inert, only comes into existence when read and assimilated[43].

Now one may replace the book by any string. Strings can be proofs of mathematical theorems in formalized language. If there is a string that proves the

Riemann hypothesis, its existence is equivalent with the truth of its statement. If there is also a string giving a counterexample, we have a proof of inconsistency. This of course assumes that it is all part of some axiomatic context. Now if there is a proof of the Riemann hypothesis there is also a shortest one. This presupposes that for each string, you need to be able to tell if it is syntactically correct, and if so whether it proves it or not. Thus you can ask questions like, is there a proof of the Riemann hypothesis which uses at most one million characters. Such a question only makes sense if you specify a particular proof checker (assuming such exist). But is there a difference between an automated proof-checker and a human? What if the shortest proof consists of 10^{10} characters. Can a human read them in a life-time, and actually understand it, make it pop up? If not does it mean that humans will never understand the proof although they would be told it is true. Would they believe it, or would it be true nevertheless, as mathematical objects, and hence truths, are objective and beyond the will of people.

What we have done is to make the spirit into flesh, reducing mathematical concepts to physical systems, and thereby making it objective and independent of man. This is what mathematical Platonism is all about, namely realism, that mathematical objects exist and truths about them, independent of man. Now what I have purported to show is of course not a real proof, mathematical reasoning can hardly prove its own existence, as little as a minimally complicated system of axioms can prove its consistency (if so we have all the reason to be suspicious as we know that inconsistent ones can). This is not mathematics but meta-mathematics, maybe even meta meta-mathematics. It is a question of faith. The truth of a mathematical proposition does not depend on who proves it, but the statement that mathematics has a Platonic reality very much depends on who claims it, just as with any statement in the humanities. Authority counts for a lot. Similarly if you prove a mathematical proposition, and then discover in the literature that it has already been proven, you are inevitably disappointed. This is because you then consider mathematics as a game, with the ulterior purpose of enhancing your status, as in any game (would you care playing a game if winning is not important to you); and if so the mathematical proposition is just a means to an end. However , if you find your arguments anticipated, concerning a philosophical question which you care a lot about, such as mathematical platonic realism, it is different. Then anticipation is reassuring rather than annoying[44].

We have so far only considered numbers, but large numbers have a lot of structure and information, and could when interpreted convey a lot of meaning. Numbers can also be very big, and the numbers we have represented by millions of digits are really rather small (and as such not representative of numbers in general). It is easy to define ordinary numbers as numbers which we can represent physically, say in a computer. Those could be thought of as numbers of zero order. Numbers of first order would be numbers that can be represented by arrays of numbers of order zero (or numbers whose digits are of order zero). Numbers of order two, are numbers represented by arrays of numbers of order one (or numbers whose digits are of order one). Now we are unable to represent numbers by arrays that large, so what does it mean really? It is easy to talk about a number with say 10^{10^6} digits (a number of order one) but we can never represent it in any physical way. Does it mean that its existence is shadier, that it is removed one more level from physically representable numbers? We may imagine a program that goes through all the representations of numbers of order one, and for each chose a number. The process would take much longer than the expected life time of the universe, but would in the end have represented a second order number. The induction process is clear, and we may speak, if somewhat loosely about numbers whose orders are numbers of first order. Or numbers whose orders are numbers whose orders are numbers of order a million. It is very hard to get an idea of those huge numbers, and their existence seem rather shadowy. In a way it is very reminiscent of the hierarchy of cardinals, and just as large cardinals never enter into mainstream mathematics, nor does explicit large numbers [45]. We may think of numbers of order zero as finite, it is within human capability to reach them by adding one at a time, while those of first order are for all intents and purposes infinite. There is an infinite number of books in the Library of Babel, as we will never in practice collect them all. For the same reason we can think of those of second order as uncountable, being in fact the cardinality of the subsets of first order, and in fact adding to the order one is like considering the subsets of the original order. To have a number whose order is of first order, is like considering the union of all power sets. If the order is of second order, it is like considering a set beyond the reach of taking the union of all power sets inductively defined. The point is that finite numbers can be very large, psychologically much bigger than infinity, which is very easy to imagine. From a strict logical point of view this is nonsense, but from a practical point of view, it is not, because the contradictions will never appear in literal applications[46]

Does infinity exist at all? It is hard to think of it physically represented by objects, on the other hand the universe may be infinite, but if so we would never know. And once you grant infinity, which is a rather simple concept after all, what about the higher ones? Then really intrepid logicians postulate inaccessible infinities, meaning ones that cannot be reached by taking power sets of power sets etc. Those are usually accepted on strong intuitive grounds, sometimes as to justify intuitions of consistency. The problem is that intuition like persuasion are subjective feelings, which cannot really be shared. The objectivity of mathematics, like that of any other science, rests on what can be shared, formulated by Popper as falsifiability. Thus the pursuit of higher cardinalities become very scholastic, and by most mathematicians seen as rather sterile. Even if the theory is logically compelling, the reasoning involved does not reach the rich interaction and profusion of ideas that characterize main stream mathematics.

What really convinces the working mathematician is the richness of the subject, the way so many different parts hang together often in unexpected ways. Your hands are tied and results are not made up but discovered, so very different from the way one may design a system of axioms for sets, an activity that shares much more with the usual administrative tasks, such as formulating laws and regulations, making up descriptive theories, in ordinary life, where conventions seem to rule, and one thing may as well be something else, as there are, unlike in mathematics, no compelling reasons. True mathematicians make up things all the time, they introduce rules as well, and make up a variety of strange concepts, seemingly picked out of a hat. And of course there is nothing logically compelling about that, and just as arbitrary as anything else we humans make up. But any invention has unintended consequences and when set free lives its own life independent of its creator. Mathematical concepts have to live and interact in a mathematical world and the way it does is not up to the whims of their creators but things to be discovered. What is this unforgiving world that kicks back at you? What is its hard matter? A natural explanation is logic, as we have discussed above, that mathematics is a big machine with moving parts acting along strict rules, just as a picture is nothing but its pixels. But to every mathematician worth his or her salt there is more to mathematics then just logical necessity. If not, we could as well just manipulate numbers, which of course is the traditional view of the mathematicians activity, and which we in a sense are doing, if everything is unsentimentally formalized and hence codified. Mathematicians are awed by unexpected connections, and just as with the rich world around us, delighted with its complexity and hidden meanings, which only gradually reveal themselves to us.

Is it all a dream? May be, but if so to what will be wake up, because what could be more real than a dream from which we cannot escape. This applies to mathematics as well as to the so called real world.

Of course this is but poetic fancy, the proof of the pudding is in the eating, and this is what the first ten chapters are meant to be.

Notes

[1]In Hyperbolic geometry we have not only models of spherical geometry, by considering spheres, but there are also so called horospheres, which are the intermediate between spheres and planes (not to be found in Euclidean and Spherical geometry), which in the same way provide models for euclidean planes within 3-dimensional Hyoerbolic geometry

[2]On a personal note. The imaginative possibility of a 4th dimension which I encountered in my early teens probably was the decisive factor in turning me into a mathematician, because it truly fascinated me. Now with the availability of various gadgets for experiencing virtual realities, one may speculate whether it would be possible to get the same feel for four dimensions as we have for three. The latter is induced in us, as noted in the introductory lectures, by our ability to move around in space, and relating muscular movement with visual changes. One may imagine two controls, both involving moving a cursor along a plane. The combined relative positions would correspond to a movement in 4-dimensional space, whose visual change could easily be computed. Would familiarity with this induce a new kind of spacial intuition? Or would the experiment have to be conducted on a newborn, which would raise serious moral concerns? Or are there biological constraints making the acquisition impossible, along the lines of if God had wanted us to fly, he would have given us wings?

[3]Thus to prove something by induction you look at the least counter-example, hoping to get a contradiction.

[4]Division with zero may be a problem. If $(x_n) \not\to 0$ means that for sufficiently big n we

have that $x_n \neq 0$ and thus with that provisio we may consider (y_n/x_n).

[5]Consider the function $f(x) = x^2 - 2$. Set $a = 1$ and $b = 2$ and hence $c = \frac{3}{2}$ we have $f(\frac{3}{2}) = \frac{1}{4} > 0$ hence we consider the interval $[1, \frac{3}{2}]$ and then $c = \frac{5}{4}$ and as $f(c) < 0$ we now consider $[\frac{5}{4}, \frac{3}{2}]$ and $c = \frac{11}{8}$ and so on. Incidentally we will get the best approximations of typ $\frac{m}{2^n}$ for each n which are by far not the best considering the size of the denominators, nor will the convergence be as quick as the one we have considered before. What it does in effect is to give the dydadic expansion of $\sqrt{2}$ according to the rule that if $f(c) < 0$ we get a 1 otherwise a 0. Another way of putting it is that if $c = \frac{m}{2^n}$ then if $f[c] < 0$ we look at the next $c = \frac{2m+1}{2^{n+1}}$ otherwise at $c = \frac{2m-1}{2^{n+1}}$. It may be worth to make a little digression. We look at the interval $[\frac{m}{2^n}, \frac{m+1}{2^n}]$ where we have $m^2 < 2^{2n+1} < (m+1)^2$ or $m^2 = 2^{2n+1} - M, (m+1)^2 = 2^{2n+1} + N$, with $M, N > 0$. We then get $(2m+1)^2 = 2m^2 + 2(m+1)^2 - 1 = 2^{2(n+1)+1} - 2M + 2N - 1$. Thus $f(c] < 0$ iff $2(N - M) - 1 < 0$, and thus the new interval will be given by $[\frac{2m+1}{2^{n+1}}, \frac{2m+2}{2^{n+1}}]$ with the new M', N' given by $M' = 2(M - N) + 1, N' = 4N$ On the other hand if $f(c) > 0$ we are looking at the interval $[\frac{2m}{2^{n+1}}, \frac{2m+1}{2^{n+1}}]$ and $M' = 4M, N' = 2(N - M) - 1$. This is easy to program. Starting with $m = 1, M = 1, N = 2$ we get the following table

	m	$m+1$	M	N
0	2	3	4	1
1	5	6	7	4
1	11	12	7	16
0	22	23	28	17
1	45	46	23	68
0	90	91	92	89
1	181	182	7	356
0	362	363	28	697
0	724	725	112	1337
0	1448	1449	448	2449
0	2896	2897	1792	4001
0	5792	5793	7168	4417
1	11585	11586	5503	17668
0	23170	23171	22012	24329
0	46340	46341	88048	4633
1	92681	92682	166831	18532
1	185363	185364	296599	74128
1	370727	370728	444943	296512
1	741455	741456	296863	1186048
0	1482910	1482911	1187452	1778369
0	2965820	2965821	4749808	1181833
1	5931641	5931642	7135951	4727332
1	11863283	11863284	4817239	18909328
0	23726566	23726567	19268956	28184177
0	47453132	47453133	77075824	17830441
1	94906265	94906266	118490767	71321764
1	189812531	189812532	94338007	285287056
0	379625062	379625063	377352028	381898097
0	759250124	759250125	1509408112	9092137

We note from this that $\frac{181}{128}$ gives a particular good approximation of $\sqrt{2}$ and thus that 1.0110101 is a very good dyadic approximation. We have $181^2 - 2 \cdot 128^2 = -7$ but as we know from Pell's equation there will be an infinite number of solutions to $p^2 - 2q^2 = \pm 1$ giving rise to the best approximations,

.

[6]One obvious way would be to renounce the existence of an inverse, we would then get a much larger concept, but of course it is the notion of an inverse which is very basic. Furthermore any group can be represented as a permutation group as Cayley, the creator of the

notion of an abstract group observed, thus in that sense making it *the* formalization of a group of permutations.

[7]The classical book by van der Waerden on Algebra was actually based on having attended lectures given by Noether.

[8]Lindemann, who had made his name proving the transcendence of π referred to the proof as 'unheimlich' which was an expression both of respect and disapproval.

[9]Let X be any countable totally ordered set. Let x_n be an enumeration, and define $\phi(x_n) = \sum_{x_m < x_n} 2^{-m}$, Then ϕ defines an order preserving injection of X into $\mathbb{R}$ (in fact to the unite interval $[0, 1]$). Now each point will be isolated to the right, by modifying adding half the value to the sum, i.e. 2^{-n+1} we achieve that every point is isolated! Anyway this construction shows that countable subsets of the reals can be extremely complicated. We may also note that if we have a countable totally ordered set such that between any $x < y$ there is z such that $x < z < y$ then it is order isomorphic with the dyadic numbers. For simplicity take a segment $[a, b]$ of the set X and let $0 \to a$ and $1 \to b$, then consider an enumeration of X with $x_0 = a, x_1 = b$ then let $\frac{1}{2} \to x_3$ as to $\frac{1}{4}$ we choose the first available x_n such that $a < x_n < x_3$ and for $\frac{3}{4}$ the first available x_n such that $x_3 < x_n < b$, and continue.

[10]Much to my surprise I have encountered people of some mathematical pretensions which have not understood it, and keep pestering you that the reasoning is flawed. It may be flawed, but if so on a much more subtle level than those people would imagine.

[11]I first encountered the proof in Gamows classic 1,2,3, infinity, which I read at thirteen, I would later return to it in other presentations.

[12]To each equation $a_n x^n + a_{n-1} x^{n-1} + \ldots a_0$ we associate the height $H = n + |a_n| + \ldots |a_0|$. To a given height H there will only be a finite number of equations. Thus we can systematically list all algebraic numbers, by ordering them by their real parts, and in the case of complex conjugates pick the one in the upper half-plane first. Thus if $H = 4$ we have $x^3, x^2 \pm x, x^2 \pm 1, \pm 2x^2, x \pm 2, \pm 2x$ incidentally only with $\pm i$ as non-rational roots.

[13]We say that a point p is a accumulation point if every neighborhood contains a point of X different from p or equivalently an infinite number, but all those points may be isolated. Not so when we consider condensation points. It is easy to see by the standard argument that any infinite subset of a closed bounded interval contains an accumulation point, that this holds for any uncountable set with no restrictions, and in fact it is easy to see that there will be an uncountable number of condensation points. Also for a condensation point p we can find arbitrarily large n such that there is an uncountable number of elements with distance between $\frac{1}{n}$ and $\frac{1}{2n}$ to p. And then we are done.

[14]Similarly we can talk about Hausdorff dimension, and it turns out to be $\frac{\log(2)}{\log(3)}$. Thus the Cantor set is somewhere intermediate between a line or a number of isolated points.

[15]There are formulas of course, but not the kind usually considered in mathematics. The Cantor set consists of the triadics only containing $0, 2$ as digits, (when there is an ending of typ $\ldots 100000 \ldots$ this is replaced by $\ldots 022222 \ldots$ as in the left endpoints of removed intervals) we get the value by replacing the 2's with 1's and considering it dyadically.

[16]

[17]There have been 44 American Presidents (not including Trump), thus there are 2^{44} different subsets of them. You can think of many ways of grouping them. Presidents born in the 19th century, Presidents dying in the 20th, married presidents, assassinated presidents, slave holding ones, red-haired ones, first names starting with A etc. Can you come up with 2^{44} different properties that characterize different subsets of Presidents? It seems a daunting proposition, one which you hardly would come up with during a life time. How should we

characterize the group that consists of Washington, Buchanan, Harding and Truman? Other than by listing their names? (or something more or less equivalent)

[18]This might be seem silly, but it is not stupid, and one may refer to Berkeley and his idealism. For people things only came into existence when thinking of them, while God was thinking of them all, all the time, in that way securing their existence.

[19]Outer measures can always be defined for sets, so V will of course have positive outer measure.

[20]Recall that a well-ordering is a totally ordered set in which every non-empty subset has a smallest member. The ordinals are well-ordered and in fact the well-ordered sets are well-ordered and counted by the ordinals, who give canonical representatives for any well-ordered set. Imagine all the countable ordinals, its union will be the first uncountable ordinal. If we have no problem imagining all the finite ordinals, and seeing clearly in our mind the accumulation, (as in Zeno's paradox) that leads to the first infinite ordinal ω, why should we stop there?
[21]C.S.Pierce in an article "On the Logic of Number". American Journal of Mathematics. 4: 8595. (1881), proceeds to prove the arithmetic properties of numbers, just along this line, being one of the pioneers of mathematical logic, anticipating Peano.

[22]Literally most of them will be unbelievably complicated, just as most numbers will be unwieldy when we try to represent them. However, just as with integers we may provide an inductive definition of what is meant by a formula.

[23]To do so is rather easy, what is remarkable is to see the point of taking the trouble to do so.

[24]In the same vein we may think of strings, specifically those made up by 0's and 1's. Given two strings s, t we may think of the catenation st (note that the order is important) which is more or less tautologically defined. We may then for the empty string denote it by 0 and then more generally define $s0' = s1, s1' = s'0$. In this way we get a model, in which the integers are economically represented by binary expansions. The inductive definitions of addition and multiplication may be used to derive the standard algorithms for such operations. Of course arithmetic operations can as well be performed on the more primitive strings $||||\dots|$ of 'sticks'. As to addition we simply do catenations, and as to multiplication we do repeated catenations. More specifically given strings s and t we string the s's after each other, one for each 'stick' we remove from t. When there are no sticks left from t we are done. Clearly the process mechanized in spirit, as all algorithms, can actually be done so in practice, by anyone sufficiently motivated to do so. But of course the literal representation is unwieldy and for most purposes unless the most trivial useless. In a real sense it has not reached the level of computation, although it is if anything a literal sense of counting.

[25]Maybe $\neg(\forall P \neg\neg P = P)$ wuld be more appropriate. But can you quantify on propositions? This illustrates the fact that formalization without a purpose is a rather pointless activity, at best only excessive pedantary

[26]Ironically what Brouwer is known for as a regular mathematician is his fixed point theorem, which is traditionally proved by contradiction (if there would be no fixed point we could construct a function that cannot exist, hence..). The theorem has applications to economics, more specifically to game theory and the existence of so called Nash equilibria. It would be rather surprising that such a mundane activity as everyday economics would depend on some rather arcane mathematical facts. What is needed are not exact fixed points, a mathematical idealization, but approximate ones with an accuracy appropriate to the situation; and economic activity as such could actually be viewed as being a process of construction of such approximations, possibly in an idealized limit process converging to a limit (never allowed to be completed). The question of the relevance of mathematical applications not only to the physical sciences but also the social, leads to an interesting and important inquiry too

intricate to be contained in this note.

[27]The distinction between potential and actual infinity is a rather subtle one. Classically mathematicians thought of infinity as potential, the actual one being a meta-physical concept, and not really the subject of a precise mathematical scrutiny. Where Cantor differed from his predecessors was to take infinity in a literal sense, actually to treat it with all the precision of a mathematical object

[28]Russell born 1872 was 42 when it was completed on the eve of the First World War. The exertion brought him to the brink of a mental breakdown and never afterwards would he be able to sustain such mental efforts. In his later work he refers to the mental effort serious philosophizing entails, that you can only sustain it for a minute at most at a time, and that the great giants of human thought (to which he modestly did not include himself) were distinguished by being able to sustain them a little bit longer. Liberated from his work he liberated himself from his first, somewhat prim wife, as well and entered a much more alluring social circle. He took to being a sophisticated but rather successful popularizer of science, philosophy and morals, doing the lecture circuits (discovering the pleasures of picking up one-night-stands en route) and becoming a public intellectual of rock-star status. This might have been his youthful ambition all along, the forays into mathematical foundations just being a digression. As to the proof of Gödel (to be discussed below), he never (along with Wittgenstein) understood it, making Gödel wonder whether they were merely pretending to be stupid or were just plain stupid. Maybe both? Sensing their stupidity they may have thought it expedient to pretend to it as to mask it. Anyway they never took up in it. Russell may have wasted his youth on such evasive pursuits, but he was rewarded by a long life, in the end, during a seemingly eternal old age, almost reaching 98, being a wonderful relict of the Victorian era, surviving into modern times in 1970. His last public display being the initiator of the so called Russell tribunal, staffed with people the likes of Sartre, assigned to reveal the war crimes of the American involvement in Vietnam. The continued career of the senior collaborator Alfred North Whitehead, was less spectacular. He suffered tragedy during the First World War losing a son in combat, (while Russell was happily having it both ways pursuing his anti-war activities), then wrote on a highly idealized philosophy, which had some impact in certain circles, but must be thought of as a philosophical current which went dry.

[29]Far into the work there is a proof that $1 + 1 + 2$. If you have not believed it before, it is hardly likely that you will be convinced afterwards, but as noted above, the purpose was not didactic.

[30]This should not be taken too seriously, technical snags tend to be eminently fixable.

[31]One may be reminded of Marx boast that he he had turned Hegel on his head. The analogy is not entirely far-fetched.

[32]Due to Barry, in the original form Richard instead talked about real numbers defined by sentences, and then used a straightforward diagonal construction to get a contradiction.

[33]The idealized computer program is a Turing machine, which inputs numbers and if eventually stopping outputting a number, according to a program which by itself is a number. Thus it is possible to design a universal Turing machine which for each number emulates the particular Turing machine, whose program is given by the number. What we are looking at is basically a set up with a Universal Turing machine which is set to emulate all the Turing machines with numbers up to and including that of the Universal plus a number to be determined whose sum is the limit. The setup is a program that involves feeding the Universal Turing machine up to the limit erasing numbers given by outputs and then looking for the smallest, outputting it. Now the size of that number does not usually grow, when we change the actual limit. With some care, it is basically a technical problem, everything will be set up. In fact it is similar to writing down the sentence ;The smallest number not expressible in X characters' and then counting the number of characters, including that of X and then substituting that for X not changing the number of characters for X.

[34]This is not strictly true.

[35]Of course the numerical representation by say decimal expansions has obvious advantages over fractions, still it presents a universal convention.

[36]If there is a tendency to perfection in evolution it is local, and often leads into *culs de sac*. It is noteworthy that when Darwin wanted to produce evidence for the historical fact of evolution he pointed to its imperfections as signs of historical contingencies. The classical vision, not shared by Darwin, that humans constitute the pinnacle of evolution, and that in fact its goal has been to produce them along a ladder of ascension, is of course disparaged by modern biologists. Evolution has no goal, and it can as well lead to degeneration as regeneration as in the evolution of parasites. In fact this popular vision of evolution was just an expression of the Genesis by other means. Evolution can as well just run around in circles, and still a large fraction of all organisms consist of primitive prokaryotes, all 'higher' forms of life are based on eukaryotes, presenting a bottleneck in the history of evolution on the planet Earth. The vision of self-improving programs is a belief in the inevitable goal of evolution as producing more and more perfect beings. In the community of enthusiasts for evolving artificial intelligence there is, in spite of the ostensibly secular and atheistic apporach, a strong element of religious faith, shared by logicians, leading to a scholastic environment. Of course outside the realm of dizzying speculation there is the option of practical implementations, denied logicians and medieval scholastics.

[37]Cf the endnote above about finite sets of American Presidents which seems only definable by an actual enumeration, so haphazard do they seem to be.
[38]Of course computers allow simulations, not only strictly numerical, and can then give evidence, just as in empirical sciences, thus strengthening the belief in the correctness of a theorem by failing to falsify it
[39]Cohen was a hot-shot in harmonic analysis, a part of the hard core of established mathematics, and expressed contempt for the feeble accomplishments of the logicians. Provocatively he asked what is the hardest unsolved problem in logic and was told about the continuum hypothesis, and he want ahead and solved it, the bastard. But logicians are mathematicians at heart, and rather to ignore his feat as being the irrelevant intrusion of an outsider embraced it (cf Hilbert's reaction to Gödel's proof), and now Cohen's 'forcing' is a favorite tool among logicians.
[40]James writes in his *magnus opus* 'Principles of Psychology' than man wants to believe, and would be willing to believe everything (as we are just too willing to believe what we read about) if we were just allowed. As organisms our beliefs are constantly being pruned by the confrontation with 'real life'.

[41]Borges is explicit in his story. Giving the number of pages, the number of lines on each page, and the number of characters on each line. For the sake of argument, say that a book consists of 500 pages, each page has 40 lines and each line 50 characters, making up one million characters. If we think of characters encoded in the standard ASCII-way we are talking about $2^{8 \cdot 10^6}$ different combinations. Each of which trivially fits on a small computer. By extending to say a thousand different characters we can include books in Russian as well, and a fair amount of mathematical formulas.

[42]If every book would occupy a volume of one liter say i.e. a thousand to the a cubic meter. We are talking about a cube of side roughly $2^{3 \cdot 10^6}$ meters. A lightyear is easily seen to be around $60'000$ astronomical units, as the speed of light is ten thousand times that of the Earth rotating around the sun. An astronomical unit is about 150 million km, or more than 2^{7+30} m. Thus a light year is about 2^{40} m. The visible universe is about $10^{10} \sim 2^{33}$ lightyears. But what is 2^{73} or even 2^{100} compared to $2^{2 \cdot 10^6}$? If we compare a cubic meter to that of the visible universe, and then make a super-universe, in which a cubic meter represent our universe. Repeat this process some 20'000 times! We can imagine how far we must travel to find books. However, if the universe would be hyperbolic, with a fairly short unit, we can imagine that we are in a circular hall say of a radius of 50 meters. Along the wall there are

256 doors, each marked by a character, going through the door, we enter a corridor the length of the same distance, at the end of which there are 256 doors, and so one. We chose a book by making the right choice of doors, in other words finding the book is equivalent to reading it. In this way we only need to walk 50 million meters, which is roughly the circumference of the Earth, a rather trivial distance, which we could do in a year or two, until we get to the last door, behind which is a book-case with only one book, the book you have chosen to get. But why walk? Why not let your fingers do the walking as it used to be written on the Yellow pages. You enter some backstreet basement office in down-town Buenos Aires, with a small sign telling you that this is indeed Borges library. You descend some steps, open a door into a small office, where a beautiful young woman sits idly by a desk painting her fingernails. She is, believe it or not, the only librarian of this huge library. She is a bit irritated as you enter as she does not want to be interrupted in her task, and asks impatiently what she can do for you. You tell her your business, and start spelling out the characters of the book you seek, sitting down on a chair. No need to walk at all. The young lady sits by a typewriter, not a laptop, as the library has not been refurbished since the death of Borges, and takes a piece of paper from a large stack by her typewriter, and diligently types all the characters you tell her. There is no hurry, you take your time, and every character takes about two seconds to type in. Thus after two million seconds, which is about three weeks, you are done, given a stack of typed papers, this is your book, which only took you three weeks to type (of course a typist may easily do ten thousand strokes in an hour, finishing a book in one hundred hours which is about four days, but you need to think), much shorter than walking around the Earth. The young woman returns to her nails, which have to be done from scratch, and as you exit, another customer with a hungry look, slips in through the door.

[43]In the notion of artificial intelligence, anything having to do with computers can be encoded as string, thus in particular an intelligence could be seen as a string, or the seed that will generate intelligences, in the vocabulary of Bostrom. But a string by itself is inert and meaningless, just like an unread book, what gives it meaning and power is how it is integrated in the infra structure. In fact this very embedding which cannot just be virtual is crucial, an aspect never given any attention by the AI-obsessed. It is this very embedding that gives it meaning. It is the same with chemical compounds, of which we may make a Babel library. By themselves they mean nothing, only how they interact with other chemical compounds give them meaning. And this will of course depend on the context, just as a given gene has no intrinsic meaning, even as a coding for making certain proteins. Its meaning only becomes clear when the phenotype is produced guided by the genetic set-up of which it is a member. Its meaning will differ from genetic context to genetic context, and it is meaningless to speak about bad genes in general, it all depends on the context. Just as one may speak about bad couples, even if there are no bad people.

[44]The same thing concerning your health. If you have arguments that 'prove' that you have a fatal disease, and then if somebody points out a flaw, and in fact you have not, rather than being annoyed at being shown a fool, you are rather relieved. The truth or falsity of the proposition means more to you, than your own cleverness in handling it.

[45]Skewes number $10^{10^{10^{34}}}$ is supposed to be the biggest explicit number that has appeared in a mathematical text, at least that was in the case of the 50's. Skewes was a South African student of LIttlewood who had shown that the function $\pi(x) - li(x)$ changed sign infinitely often, and was asked to get an effective upper bound for the smallest number for which it was strictly positive. Now $\pi(x)$ denotes the number of primes $\leq x$ and $li(x) = \int_0^x \frac{dt}{\log t}$. In this upper bound, assuming the Riemann hypothesis, 34 should be replaced by 964 if not, has subsequently been improved. In many *a priori* estimates one has a double exponential bound, here we have a triply exponential.

[46]Of course we all have the experience of getting to the limit when we want to add and multiply big numbers, and would like to extend our range of ordinary numbers a little bit more, but by modest operations we would never reach typical numbers of the first order. Now we can represent very large numbers by recursive definitions. We start out with addition $m+n$

then multiplication as repeated addition $m \cdot n = mn$, then powers as repeated multiplication $m * n = m^n$. I it becomes tedious to invent new symbols all the time, why not index them by numbers and define inductively $F(0,m,n) = m+n, F(1,m,n+1) = F(1,m,n)+m(= mn+m = m(n+1)), F(2,m,n+1) = F(2,m,n)m(= m \cdot m^n = m^{n+1}$ or more generally $F(k+1,m,n+1) = F(k,F(k+1,m,n),m)$ and to make it well-defined we need to specify $F(k,m,0)$ we have $F(0,m,0) = 0, F(1,m,0) = 0, F(2,m,0) = 1, F(3,m,0) = m$. It gets a bit complicated but gives us some idea. A simpler way would be to start out with a seed $\phi(0,m) = 2^m$ say (although defining $\phi(0,m+1) = \phi(0,m)^{\phi(0,m)}$ with $\phi(0,0) = 2$ would give even more juice, but rather pointless) and define inductively $\phi(k+1,m) = \phi(k,\phi(k,m))$ The inductive procedure allows us eventually to get $k = 0$ in the first argument. Thus $\phi(2,m) = \phi(1,\phi(1,m)) = \phi(0,\phi(0,\phi(1,m))) = \phi(0,\phi(0,\phi(0,\phi(0,m)))) = 2^{2^{2^{m}}}$. The Peano axioms may it all well-defined, but what is say $\phi(10^6,1)$? One may note that these simple ideas, rediscovered by many a high-school student, go back to Wilhelm Ackermann (1896-1962). The ideas have many variations, and the simplest formulation known as the Ackermann-Peter function is defined as $A(0,n) = n+1, A(m+1,0) = A(m,1), A(m+1,n+1) = A(m,A(m+1,n))$. Ackermann presented them as examples of totally computable functions, but not so called primitive recursive, as they grow too fast.