Second Lecture

Euclid

Not much is known about Euclid, not even his span of life, although by indirect means one may conclude that he was active around 300 B.C.

His main achievment is his not so much his compilation of mathematical results at the time, but as the masterly way he presented them, providing an example to not only all mathematical but also scientific works ever since. It cannot be emphasized enough that the main asset of the deductive method is not so much the establishment of proof (beyond any reasonable doubt), which is a meta-physical statement; but that the method makes for transparency and hence simplifies criticism, which is the basis for all scientific endeavor. In so doing, it makes subterfuge difficult and encourages honesty. There are also some pedagogical advantages to the method, although those tend to be exaggarated. There is much more to mathematics than deductive reasoning.

Two anecdotes are ascribed to Euclid, and whether they are correct or not we can of course never ascertain, and hence that is of minor importance, what is important is the light it throws on the underlying attitude.

The first concerns the statement of Euclid to the effect that there is no Royal Road to Geometry, as a response to some local illuminary, who daunted by the mass of material to be mastered, asked for a short cut. The moral is that to a King there are many short-cuts and privileges, but none that applies to study. When it comes to mathematics we are all equal, (just as we are when it comes to God).

The second concerns a student, who after painfully learning his first theorem, asked Euclid by what he could gain from it. This is a common query proposed by many a student to the exasperation of their teachers. Euclid called on his slave, asking him to give the man a coin, because evidently he needs to gain something from what he learns. The moral is of course that study brings its own award.

Those are two very useful things to keep in mind when studying mathematics.

The Books of Euclid's Elements

There are thirteen books of Euclid's Elements. They constitute a compendium of classical Greek mathematical knowledge. But Euclid was no mere compiler, he took great care in making a clear presentation, selecting a small set of axioms and postulates, and proceeding carefully and without any real mistakes to produce a systematic chain of deductions going from the simple to the more complex. It is true that Euclid assumed principles of reasoning and assumptions about space (such as the possibility of rigid motion) that he does not make explicit, but the main thing is that there are no wrong theorems in Euclid. It can still be used as a text-book, and still versions of Euclid form the core school-curriculum in geometry in large parts of the world, and did so in Sweden until the 60's. Euclidean geometry was thus the first place most people encountered deductive mathematics, an experience that can be so momentous as to overshadow the actual geometrical contents which are rather limited in scope.

One should also keep in mind that Euclid wrote other works as well. There is work on optics as well as conics and works which have been lost.

We will now list the contents of Euclids thirteen books with short commentaries and then consider in more detail the theory of magnitudes, Pythagoras theorem, and solid geometry.

The mathematical content of Euclid

Book I

Preliminaries (Definitions ($o\rho o\iota$), Postulates ($\alpha\iota\tau\eta\mu\alpha\tau\alpha$) and Common Notions (axioms) ($\kappa o\iota\nu\alpha\iota\epsilon\nu\nu o\iota\alpha\iota$)

Among the Postulates, the notorious number five.

Constructions of triangles, parallelograms

Propositions, concerning congruences of triangles, their angular sums, Pythagoras

Book II

Geometrical algebra ($a(b+c+d+\ldots) = ab+ac+ad+\ldots$)

Geometric solutions of quadratic equations

Construction of squares with equal area to that of a given rectilinear figure.

Book III

Propositions concerning circles, tangents, secants, central and inscribed angles

Book IV

Inscribed and circumscribed circles, constructions of regular pentagons, hexagons and the 15-gon.

Book V

The theory of proportion between magnitudes

(If $a : b = c : d$ then $a : b = (a+c) : (b+d)$)

Book VI

Similar figures

Proposition: The areas of similar triangles areto oneanother in the duplicate ratio of the corresponding sides

Book VII

Basic definitions of divisibility, primes, divors etc

Proposition. If a prime divides (measures) a product, it will divide one of the factors.

Book VIII

Geometrical progressions

Book IX

Proposition. The infinitude of primes (to any finite collection of primes we may find a prime not contained in it)

Book X

The classification of Incommensurables

Expressions of type $\sqrt{\sqrt{a} \pm \sqrt{b}}$

Book XI

Basic definitions of solid geometry

Book XII

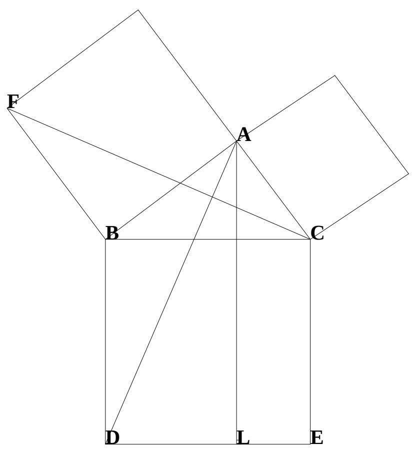
Computations of areas and volumes. The method of exhaustion. Areas of circles proportional to the squares of their diameters. A cone has the third of a volume of a cylinder with the same base and height.

Book XIII

The Platonic Solids.

Proofs of Pythagoras

Euclid gave the following proof of Pythagoras. It is very much in the spirit of Euclids standard proofs, drawing ancillary lines, looking for congruent triangles, identifying further lengths and angles, and thus climbing up a ladder, one step reached by the hand becoming the foothold for the next. Very much in the spirit of a deductive journey through mathematics.

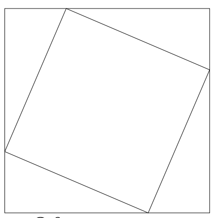


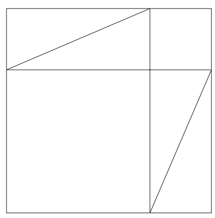
A right-angled triangle (with its hypotenuse as its base) is complemented by the squares on each of its sides. We draw three auxiliary lines, two of them from the crucial point A located at the right-angle of our triangle, and one joining F and C. The first two connects either A with B or drops perpendicularly onto the side DE at L. The triangles ABD and FBC are congruent, as the sides AB and FB are equal by construction, as well as the sides BD and BC, while the respective angles at B are equal (the angle of the triangle ABC adding a right one). The area of the triangle ABD is half of the area of the rectangle passing through BDL, while the area of the triangle FBC is likewise half of the square passing through ABF, thus the rectangle BDL has the same area as the square FBC. The argument could as well have been used on the other square on the right, whose area would than have turned out to be that of the rectangle through CEL. Thus the area of the square of the hypotenuse is equal to the sum of the squares of the two other sides.

It is impeccable, and very clever. Notice how he dissects the area of the bigger square and redistributing it to the two smaller ones. Note that Euclid does not give a formal definition of area, nor prescribes an explicit method of its computation. All what he does is to subject the notion of area to certain principles, such as it being conserved by movement, that a subset always has smaller area than the set which contains it, and that the area is additive on disjoint subsets. One can think of this as an instrumental definition[1]. Areas can be manipulated in certain ways, and in practice this means that they are cut up and reassembled. Implicit in the proof above is a rather elaborate divison and reshuffling of the large square into triangles. Can you explicate the actual process? This method allows a direct comparison between areas, which are special types of magnitudes. But it does not guarantee that the process will always be succesful. There is very hard to come up with an explicit area that matches that of a circle. I.e. squaring the circle, meaning literally finding a square with the same area as that of a circle. This terminology has survived until today, when we speak about squaring an area or more generally an integral, although the translation of 'quadrature' is more common in the context.

Now the proof above is impeccable, but it has some drawbacks. First, it merely verifies that something is true, it does not 'explain' why. The steps that are taken are clever, but not obvious and compelling, and you wonder how Euclid could have come up with them. After seeing the proof you may nod, but you may quite likely forget about it after you have seen it.

There are maybe a hundred different kinds of proofs of Pythagoras. They are not really all different but many are just variations of each other. You may have come across other proofs. This is one, which may be the most popular. It is best presented by two figures





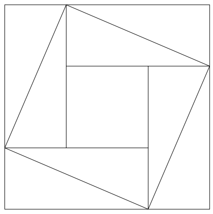
Often it is expressed as

$$(a+b)^2 = a^2 + 2ab + b^2 = c^2 + 2ab$$

where the areas of the triangles are given by $\frac{ab}{2}$, but this algebraic interpretation is of course not necessary, in fact the geometry explains the algebra.

This is a simpler proof and much more likely to stick in your memory. In a way it is similar to the previous proof that it involves a rearrangement and a subdivision, but this is done in a more clever way and involves an addition of areas as well as a division and reshuffling.

A variation on the theme is given by the following figure



Illustrating both

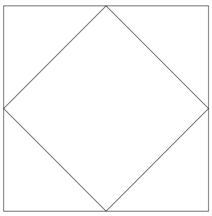
$$(a+b)^2 - (a-b)^2 = 4ab$$

and

$$c^2 - 2ab = (a-b)^2 = a^2 + b^2 - 2ab$$

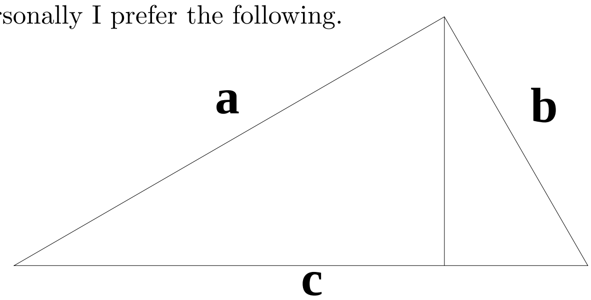
from which the result also follows by an algebraic manipulation, although it is not quite as easy to see that $(a-b)^2 = a^2 + b^2 - 2ab$ visually.

Now a degenerate case is given below.



Note that the smaller square is easily seen to have half the area of the larger. As areas scales as squares we see that the sides of the two squares are like $\sqrt{2} : 1$. Thus we do not need to use Pythagoras theorem to find the length of the diagonal in a (unit) square, we only need to know the basic scaling principle.

Are there even simpler proofs? In fact the remark above gives the clue. Personally I prefer the following.



It depends on two basic facts. One that areas scales as squares of the linear dimensions (of a triangle), the second that a right-angled triangle can be decomposed in similar triangles as in the figure.

The last is easy to see by inspection, while the first requires some more careful proof. Given them we conclude that the area of each triangle is proportional to the square of the hypothenus. As areas add we can write

$$kc^2 = ka^2 + kb^2 = k(a^2 + b^2) \Rightarrow c^2 = a^2 + b^2$$

where k is the constant of the proportions, varying by the shape of the triangle. Its introduction is a bit ugly, and we will see how the theory of proportions as presented by Euclid, gives a more elegant formulation. Of course the first statement has to be proved, and we will do that later, but nevertheless in my mind this is the simplest proof of Pythagoras. If you have seen it once, it is hard to forget it. It also, somehow, gets a bit closer to giving an understanding why things have to be as they are as opposed to just giving a verification of a curious fact. Pythagoras theorem is elementary but it is important. It is used as a defintion of length in the context of Cartesian co-ordinates, and more abstractly as the prototype of the theory of quadratic forms. An Euclidean space is as you know, a real vector space equipped with a positive definte form. Thus the significance of a result or a concept cannot be judged in isolation but can only be inferred from the various contexts it will appear.

The theory of Proportions of Euxodus

Magnitudes are not numbers. The distinction has become blurred in modern education as we tend to think of the number line, in which the integers, both postive and negative are embedded. Thus if you ask students whether $1 + 1 = 2$ is really true, it happens that they come up with the suggestion that it might perhaps only be approximately true, that in fact if we 'measure' things more accurately we may come up with something like $2.000'000'000'1\ldots$. This is clearly a so called category mistake. Numbers and magnitudes belong to different categories. Any identification between magnitudes and numbers are based on a tacit unit. In Euclidean Geometry there are no mathematically defined units. If you want to produce a unit, you have to point at one.

> It is typical that until rather recently the standard of length, i.e. the meter, resided as a rod of platinum kept at a constant temperature in some basement in Paris. A unit thus being a physical object that cannot have a mathematical definition, it has to be pointed at. However, the original defintion of the meter was more mathematical, if more impractical, namely it was defined as a 40 millonth of the length of the equator[2]. The significance of which, will be become more clear when we later on will discuss spherical geometry. Still there is a physical prototype, not mathematically defined - namely the earth. However, the definition has the advantage that different civilizations of the earth could communicate the definition without being in physical contact with each other and thus unable to point at common specific things (except of course the earth itself of course). In particular such a definition would enable us to transmit definition over historical time when actual physical prototypes may have degenerated or got lost (it is hard to lose the earth itself!).

The theory of magnitudes is a sophisticated theory and in many ways quite modern. It is not due to Euclid but goes back to Eudoxos. Magnitude is a basic and intuitive concept which is hard to define in any reasonable way, and the definition which Euclide supplies is inevitably circular as are his definitions of point and line. However, two fundamental features evolve. One concerns comparability. Two magnitudes are comparable if either is contained in a multiple of the other. Thus there are different kinds of magnitudes. Magnitudes of lengths are not the same as magnitudes of areas or volumes, to say nothing about weight and time. You never add things of different magnitudes, just as you do not add apples and pears, unless you reduce them to a common feature (such as say pieces of fruits). You may multiply magnitudes, but then they become different magnitudes, just as the product of two lengths is not a length but an area, represented say by the area of the rectangle with the given lengths as sides. Furthermore two things may be comparable but not commensurable. The latter means that we may find a common unit, of which each is a multiple, or equivalent, we may find two common multiples. All integers are commensurable. Two integers can always be measured by some common unit, meaning, expressed as multiples of some common divisor, which natu rally is taken to be the largest one, similarly any two integers have common multiples. As noted before this does not hold for magnitudes of lengths that occur naturally in geometry, such as those given by the side and the diagonal of a square. As the Pythagoreans realized there is no common unit to the two, the quotient whatever it may be, cannot be represented as a qoutient of integers. It is, as we say nowadays, irrational.

> To call the ratio $\sqrt{2}$ is a description, and does not solve the probles. What is $\sqrt{2}+\sqrt{3}$? it would then be a number whose description is not simpler as the question it is to answer.

The Greeks wanted at all costs to avoid dealing with irrational numbers, and the reasons for so doing were fully rational. They wanted to argue rigorously.

The crucial problem is to be able to say when the quotient of two magnitudes are equal to the quotients of two other magnitudes. Or as they preferred to put it, when two pairs of magnitudes are proportional to each other. A typical example is the case of two similar triangles (i.e. triangles with congruent angles), when corresponding sides are proportional to each other.

So let us denote the mangitudes by a, b and c, d say. The proportions a/b and c/d being equal is equivalent to the following statements.

i) Whenever $na > mb$ then $nc > md$

ii) $na = mb$ then $nc = md$

iii) $na < mb$ then $nc < md$

First implict in the definition is that a, b and c, d have to be comparable. Thus you cannot divide magnitudes of different kinds (i.e. non-comparable). Secondly note that in our terminology i) means that if $a/b > m/n$ then $c/d > m/n$ while iii) means that $a/b < m/n \Rightarrow c/d < m/n$ and that ii) may only occur if a, b are commensurable, and if so c, d need to be commensurable as well. In fact if a, b are commensurable, ii) implies both i) and iii). We note also that by only requiring i) we can say that a proportion a/b is less than a proportion c/d. The whole thing is very analogous to the definition of the reals by Dededkind using cuts introduced in the 19th century.

> Recall that a cut is a subset of the rationals such that if x is in the cut and $y > x$ then y is in the cut. Examples of cuts are $x > a$ or $x \geq a$ for

some rational number (or $-\infty$), but not all cuts are of this simple form. The cut defined by $x > 0 \wedge x^2 > 2$ is notoriusly not of this form. It is easy to define an arithmetic on cuts (although not without some technical problems when you consider subtraction between cuts necessitating left and right cuts) and thus get a definition of real numbers endowing it with the property that any right or left cut bounded to the left and right respectively has a smallest and biggest number, so called greatest lower bound or least upper bound respectively.

Another implicit notion is the action of the integers on the magnitude. For any integer m and magnitude a we can speak about ma. This will satify rules such as $m(na) = (mn)a$ as well as $(m+n)a = ma + na$ and $m(a+b) = ma + mb$ and Euclid extends this to multiplication and addition of magnitudes, writing down formulas such as

$$a(b + c + \ldots) = ab + ac + \ldots \quad (a+b)c = ac + bc$$

known as geometrical algebra, and easily illustrated geometrically.

Euclid is careful, he shows that equality between proportions is an equivalence relationship, and that we have

$$a/b = c/d = (a+c)/(b+d) \quad a/b = c/d \Rightarrow (a+b)/b = (c+d)d$$

or to use a less compromised notation

$$a : b = c : d = (a+c) : (b+d) \quad a : b = c : d \Rightarrow (a+b) : b = (c+d)d$$

where $a : b$ should not be thought of as a number but an entity for which have the notion of (total) ordering.

However the definition does not allow us to divide magnitudes of different types. Thus we would not directly being able to divide length with time and speak about velocity, but would express two velocities as being equal if distances covered are proportional to times spent. Now we can note that $a/b = c/d$ is equivalent to $a/c = b/d$ when we think of the proportions as numbers, which suggests that we would have an alternate criteria for equality of proportions between different types of magnitudes by this trick. Or more explicitly

i) Whenever $na > mc$ then $nb > md$

ii) $na = mc$ then $nb = md$

iii) $na < mc$ then $nd < md$

Tempting as it is to propose that, it is quite another thing to find documentary evidence for it. Still it is felt as being legitimate to say that this would lie entirely within the conceptional understanding of the Greek.

So let us try and prove within this theory, although not necessarily the way Euclid did, that areas scale as squares (this is proposition 19 of book VI). Recall that the Greek did not think of quotients $\frac{a}{b}$ as numbers, and thus not $\frac{m}{n} > \frac{a}{b}$ as expressing that the rational number $\frac{m}{n}$ is larger than the number $\frac{a}{b}$, but instead writing $mb > na$ as saying that the proportion $m : n$ was greater than the proportion $a : b$. To prove the proposition we need to prove a lemma to the effect that for any rational number $\frac{m}{n}$ there is a rational square $\frac{p^2}{q^2}$ less (greater) than it, such that the difference $\frac{m}{n} - \frac{p^2}{q^2}$ is arbitrarily small. This can be done in any number of ways, and best left to the reader[3]. Now let us proceed. Given two triangles Δ, Δ' with corresponding sides a, a'. Choose m, n such that $ma^2 > n(a')^2$, we have proved that we can find p, q such that $p^2a^2 > q^2(a')^2$ from which we conclude that $pa > qa'$. But then denoting by $A(*)$ the area of a triangle (*) we have that $A(p\Delta) > A(q\Delta')$ which can be written $p^2A(\Delta) > q^2A(\Delta')$ which translates into $mA(\Delta) > nA(\Delta')$. Similarly for inequalities in the other direction, and we are done.

So let us formulate the last proof presented above of Pythagoras. Having disected the right-angled triangle into two similar triangles, each similarto the whole. We note the proportionalites $a^2 : A, b^2 : B$ and $c^2 : C$ from which we find the proportionality $(a^2 + b^2) : (A + B) = (a^2 + b^2) : C$ as areas add. From the last we get the equality $c^2 = a^2 + b^2$. Note that we do not have to introduce some dummy proportionality constant.

Number theory

We should not forget that Euclid did not only deal with geometry but also with numbers. Any two integers are commensurable, as they have 1 as a common unit. But numbers can be measured by other units as well. In fact for any divisor d of a number n, we can think of n measured by d in terms of n/d copies. Given any two numbers m, n the natural problem is to find the biggest unit d, with which both can be meausured. Intimately connected to this is to find the smallest common multiple. Recall that two magnitudes a, b (of the same type) are commensurable if we can find integers m, n such that $ma = nb$. This problem was solved inductively through what has become known as the Euclidean algorithm. Given two numbers $m > n$ then we can write $m = kn + r$ where $0 \leq r < n$ by successively subtracting n-units from m. If $r = 0$ we are done. We see that n is the greatest common divisor to m, n and also that m is the smallest common multiple. Now define $L(m, n)$ as the largest common divisor. This will satisfy

- i) $L(m.n)$ divides both m and n
- ii) If d divides both m and n then d divids $L(m, n)$

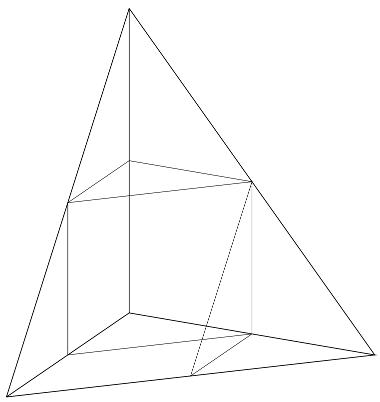
Note that we do not yet know that such a number exist, but it follows from the division algorithm. In fact we see that $L(m, n)$ exists iff $L(n, r)$ exists and if so they are equal. Note that dually we can define the smallest common multiple $M(m, n)$ and that exists iff $M(n, r)$ exists and if so they are equal. In fact if d divides both m, n we see that d divides r, conversely if d divides both n and r

it divides m[4]. If $r = 0$ then as we have seen both $L(m, n)$ and $M(m, n)$ exist, and we can go backwards. As the successive remainders $r = r_1 > r_2 \ldots$ form a strictly decreasing sequence, the case $r = 0$ will be reached in a finite number of steps. We can also note that r is a linear integral combination of m, n, this will then hold inductively for $r_2, r_3 \ldots$ and thus that $L(m, n)$ can be written as a linear integral combination of m, n. Particulary interesting will be pairs of numbers m, n with $L(m, n) = 1$. Those have no common unit except the obvious one. Furthermore numbers p such that $L(p, m)$ always is 1 or p are special, and are of course the primes. By disregarding 1 we see that no two distinct prime numbers have a common prime factor. The proof that there is an infinitude of primes is one of the most celebrated theorems of Euclid. Euclid does not express it in this way, but rather given any finite collection of primes $p_1, P_2 \ldots p_n$ we can cook up a new prime p distinct from those given. We need a lemma to the effect that any number n is divisible by a prime. The argument is similar to the Euclidean algorithm, in that we exhibit a sequence of numbers $n = d_1 > d_2 \ldots$ such that d_{k+1} is a proper divisor of d_k. This sequence cannot be continued beyond d_k iff d_k has no proper divisors (i.e. is a prime)[5]. We can now write down the number $p_1 p_2 \ldots p_n + 1$. This is not divisible by any of the p_i's, Note that we do not need unique factorization for this. Thus any of its prime divisors need to be different from the given primes. Euclid then goes on to show unique factorization, the crucial element of which is to show that if a prime p divides a product, it needs to divide at least one of its factors.

What is to be noted that any elementary introduction to number theory cannot be improved on Euclid.

Solid Geometry

Much of the chapter proofs what we now would express as formulas for the volumes of various solids, such as cylinders and pyramides. Euclid would instead express them in terms of proportions. Cylinders and pyramides of the same heights have volumes proportional to the areas of their bases, and with fixed bases they have volumes proportional to their heights. In modern terminology the volume of a cylinder or a pyramid, including that of a cone, would be of the form kBh where B is the area of the base, h the height and k a constant of proportionality. It may not be so hard to figure out k for the case of a cylinder, but to find it for, and hence the formula for the volume of, a pyramid is a bit subtler, as unlike the case of triangles, you cannot combine them into something (such as a rectangle) whose area is obvious. However, there are ways of getting around it, as long as you know the principle that volumes scales like cubes. In particular look at a tetrahedron below.



By slicing the tetrahedron halfway between two of its faces, we get a decomposition into two similar tetrahedra and two prisms. The prisms share faces of the small tetrahedra (but not the same) and with corresponding heights the same. The volume (V) of a prism is given by the area of the base times the height, and the volume of a tetrahedron is proportional to both the area of base and the height, thus it is given by kV where the proportionality constant has to be determined[6]. The big tetrahedron will have volume $8kV$ thus we get $8kV = 2V + 2kV$ from which we get $6kV = 2V$ i.e. $k = 1/3$

This is not the way the Greeks derived the volume of a pyramide, or more generally a cone. They could have used the method of 'exhaustion' later to be employed with such ingenuity by Archimedes. They would have reasoned something like this. Let P be the volume of the solid formed by the two prisms. In each of the two smaller tetrahedra, we can find scaled down models of the prisms, each of which will have volume $\frac{1}{8}P$ by scaling. There will be two of them. At the next steps there will be four and eight etc. Thus we get a geometric series of subsequent volumes

$$P + \frac{1}{4}P + \frac{1}{16}P + \frac{1}{64}P + \dots$$

Those finite sums come arbitrarily close to the volume of the Tetrahedron, in fact they exhaust them. Now, we would be tempted just to sum the infinite geometric series which is elementary enough and call that the volume. The Greeks would be far more careful, arguing in fact that any volume differing from the sum, would lead to a contradiction. (We will return to this in the next lecture on Archimedes). Anyway, the volume will be given by $\frac{4}{3}P$. Now the volume of anyone of the prisms is given by the area $\frac{1}{4}A$ of its face times the length $\frac{1}{2}H$ of the corresponding height. Thus in toto $P = \frac{1}{4}AH$ (remember that there are two prisms). For the tetrahedron, the area of the face will be A and the corresponding height H hence its volume will be $\frac{1}{3}AH$.

The previous argument is clearly nothing else but in disguise the clever way of computing the sum of an infinite series.

The last books of Euclid deals with the Platonic Solids, showing how to construct them as well as showing that there cannot be any other regular polyhedra save those five known since antiquity[7]. To compute the radi of circumscribed, inscribed and intermediate spheres (spheres passing through the vertices, tangent to the faces and edges respectively) would have been within their technical ability, as well as computing their surface areas and volumes.

Notes

[1]The crucial thing is that the areas of triangles with the same base and heights of the same magnitude are equal. If the height drops onto the base it is straightforward, otherwise you will have to consider differences between areas.

[2]In other words a great circle, but by the time the meter was defined it had been established that the Earth is not a sphere but a rotational ellipsoid, with the polar diameter significantly shorter ($\frac{1}{300}$) then the equatorial. The meter was then defined as one 10^{-7} of a meridan from the equator to the pole.

[3]Here is a suggestion. Write $\frac{m}{n} = \frac{k^2m^2}{k^2mn}$ and let $p = km$ and q^2 the smallest square bigger than k^2mn, then $0 \leq \frac{m}{n} - \frac{p^2}{q^2} < (\frac{m}{n})\frac{2q+1}{k^2mn}$ where $(q-1)^2 \leq k^2mn$ and from this we conclude the estimate $\frac{m}{n}(\frac{2}{k} + \frac{3}{k^2})$ (using the inequality $l \leq l^2$ for integers) that if k is big enough the difference can be made arbitrarily small.

[4]If $m = ad, n = bd$ then $r = ad - kbd = (a - kb)d$. If $r = cd, n = bd$ then $m = kbd + cd = (kb + c)d$

[5]A slicker argument would be to consider the smallest number that has no prime divisor. It cannot be a prime, hence it has proper divisors, which by definition has to have prime divisors.

[6]One should note that the cubic scaling of volumes cannot be as straightforwardly proved as in the plane case, because a tetrahedron cannot be decomposed into similar tetrahedra. But this is of course the case for other solids such as cubes, and this allows a round-about way to prove the general case by a method of exhaustion.

[7]There are strong archeological evidence that they have been known since pre-historic times.