Fourth Lecture

The Decline of Greek Civilization and Mathematics

As we have seen Classical Greek culture and Civilization survived the world conquering ambitions of Alexander. However, its center of gravity diffused to the periphery, while mainland Greece played a subordinate role[1]. The center of Greek learning became Alexandria, while Archimedes was active in Sicily. This was not a big deal, Greek civilization had also during its heyday spread by colonies all over the Eastern Meditteranean and what is now known as the Middle East. It was a civilization based on trade not on political empire, although the rivalry with the Persian empire temporarily brought about a political unity.

What killed Greek Civilization? First it was the Romans, they conquered Greek and subsumd its lands into the hegemony of the Roman Empire which at its height was unrivaled, with Persia, India and China being the only civilizations beyond its domination.

When Caesar set the fleet of the Ptolemy dynasti on fire in Alexandria, the conflagoration spread to the city and burned down the library. This was a disaster, but not unmitigated, as copies of work survived at other location and soon after that much of the library was restored.

The Roman civilization was a practical one and had no interest in mathematics per se. During the Roman period no progress in mathematics was made. It was not that the Romans were indifferent to Greek culture, on the contrary it was very much held in regard, and much of Roman high culture was an attempt at emulation. To be cultured in Roman times meant knowing Greek on the other hand Latin developed as a language, partly I suspect on Greek models and inspiration. There were translations of Euclid and other Greek works in mathematics, but perfunctory. There simply was no interest. It is remarkable that such a developed civilization as the Roman showed no interest in mathematics during the several centuries it was in domination. Why is that?

The next disaster that befell Greek culture was Christianity. The Christians a persecuted minority became ruthless supressors once in power. Greek culture was pagan and should be stamped out, and there was actual wilful destruction of it. The Christians were not concerned with the here and now, they were focused on the afterlife. As noted the very opposite of what the Romans, they were supplanting, stood for. Both attitudes seem not to be conducive to the cultivation of mathematics. Christianity is, unless of course you are a believing Christian, a hodge-podge of ancient religions grafted on the monotheistic stem of the Jewish one. In particular it shows traces of many cultural influences, including even those of the Greeks. Neo-Platonism is one. Platos heaven of eternal timeless forms as opposed to the low sensual world of the senses is very congenial to Christian ontology. A name to remember is St Augustine (354 - 430 A.D), a very clever and learned man, not to say a truly remarkable one, who as a theologian[2] deeply influenced Western Christianity and among other things anticipated the scholastics some eight hundred years later[3]. Still

Christian morality is very much with us, regardless whether we are believers or not, and hence there is a chasm that divides us from Pagan Greek culture. A chasm most people are not really aware of. We do often stress our continuity with ancient Greece through the concept of Democracy. However, Democracy meant rather different things to the Old Greeks than it does today. To the Greeks it was, as noted in a previous lecture, the tradition of free inquiry and critical discussion that was important, while we, at least in populistic terms, tend to confuse Democracy with egalitarianism. A classical Christian tradition is that of humility, which would have been strange to the Greek[4].

Recall that the ascendancy of Christianity coincided with the rise of the emperor Constantine, and it was also very mixed up with the decline of the Roman Empire. We are talking about early 300 A.D. when Constantin founded Constatinople (330 A.D). The split of the Roman empire in the west and the east occurred in the next century, and soon thereafter (476 A.D.) the Western Empire disintegrated from the onslaught of barbaric tribes. This was a period of great unrest on the European peninsula with large migrations of people. With the disintegration of the Western Roman Empire we have the onset of the Dark Age, while the Eastern Roman Empire would survive for almost another thousand years until 1453 with the fall of Constantinople, a date to which we will return. Of course the Dark Age did not come suddenly, although it is possible to date it politically. The spread of Christianity certainly was very instrumental in bringing it about.

The death knell of the Greek culture was delivered by the Arabs. As you all know starting with the death of Muhammad (570-632) the Muslims were bent upon world conquest and were singularly successful, in fact bringing this about in a few decades, what had taken the Romans centuries. They conquered the Middle East and North Africa, the heartland of Christianity. Alexandria was sacked (in 642), the books in the precious library heating the baths of that city for a few years. As the rationale was if it is useful it is already in the Koran, and thus superfluous. If it is not, it is harmful and should be burnt. Similar sentiments were expressed by Augustinus[5]. There is nothing particular dogmatic about Islam that was not shared by the Christians. Both religions are strikingly similar (Islam being a purer and more radical monotheistic version of Christianity more divorced from Pagan influences) thus this common mistrust.

The final destruction of the Alexandrian Library was indeed a disaster. We can only speculate about what was lost. This was a time when books tended to be like works of arts are nowadays, unique artefacts, existing typically in no copies. In classical literature there are many references to works, which if not entirely lost only survives in fragments. One can only speculate as to works which are lost along with their references and hence conclusively erased from the historical record. Could there have been philosophers on the status of say Plato who are now irretrievably lost? We can only speculate. History is a reconstruction of the past using the available sources. When sources are gone, any such reconstruction is bound to be partial, maybe even fragmentary, and thus possibly very misleading. About this we can only accept. One thing is sure, the Greek culture was very close to have been obliterated in historical memory.

It was not, however. But it shows the fragility of civilization over time.

The Arabs were extending their territory, conquering Southern Italy and Spain. Its northward expanse was halted in France by Martell in 732 and by the presence of the Eastern Roman Empire, yet still strong enough to muster effective resistance. After a period of feverish expansion, Arab civilization started to grow and flourish, but that is another matter.

You may wonder why I digressed on religion? This will become apparent later on I hope. Religions are perhaps the strongest cultural manifestations humans are capable of, and there are strong relations between mathematics and religion, to which I plan to return.

Basic mathematics - counting

One striking thing about Greek mathematics is its sophistication. As such we see it as very modern. To some extent this is of course almost tautological. Greek mathematics has been the inspiration of all Western Mathematics and hence of all Mathematics worth its name. No other mathematics developed by other civilizations have reached this level by far. Read a Greek text linguistically translated and you have no difficulty understanding what it is all about and to follow the arguments. Read a corresponding text in Hindu or Chinese, and only a most sympathetic reading will make sense of it. Is mathematics not universal, and if so, should not mathematics emerge in all civilizations (in particular as mathematics from the Platonic point of view is timeless and independant of human whims)? This is a very interesting question, and to many maybe the most interesting as when it comes to perspectives on mathematics.

Why do we need mathematics? To start from the most elemental beginnings we have the issue of counting. When it comes to subitizing, humans are suppsedly no better than rats and crows. What I am talking about is assessing numbers without counting. Some say that the limit is four or five, I believe it is more like three. We do have very limited cognitive abilities, as I noted on par with other mammals and with birds. In fact in some specialized pursuits animals have abilities superior to humans[6]. However, we are able to transcend our limited capacities by counting. What is counting? The origin of counting is language. This is of course obvious when we think of names for numbers. Language is one example, and maybe the most important, when it comes to so called extended cognition. Language allows us to create an external tool, whose mental origin, should not mislead us as to its externality. But there are other, perhaps more obvious ways, of extending our cognition by creating physical artefacts, such as pebbles, mark on paper or clay etc. When it comes to names for numbers we can of course not make up independant names for each integer. It certainly would be far too complicated to remember, especially taking into account our feeble cognitive powers (there is no reason the memory of a horse or an elephant should be inferior to that of ours, on the contrary). Thus by necessity there will be some system of structuring and construting numbers. The number 1641 say is more of a sentence than a word, adhering to some syntax. I believe that the linguistic representation often was more sophisticated than the standard symbolic.

Why do we need to count? We take it for granted and many years of elementary school is dedicated to it[7]. In fact for most pupils mathematics taught at elementary school is a chore, and as such painful, regardless of innate ability. When in practical life do we need to count? Are we in greater need of it now than in the distant past?

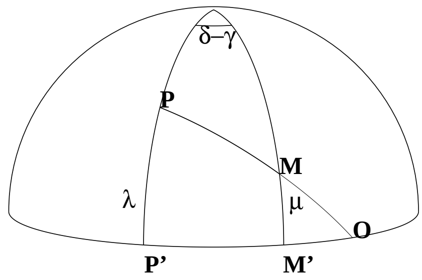
What kind of things do we count? Ultimately could it be that most of the counting that goes on in life of an economic nature? Trade and commerce. The counting of goods and the counting of money. Not to count correctly in business transactions is a form of cheating. Thus all civilizations are in need of procedures for counting. Counting should be a skill, although based on understanding eventually independent of it. We see here the germ of the important notion of method.

The Greek had an inferior counting system. And the Roman way of symbolically representing numbers is very quaint and has survived for that very reason into modern times. It is generally agreed that our positional system of representing numbers is due to the Hindus, especially that of the invention of the zero, although there has been plenty of predecessors, such as the Babylonian system. However, an idea not fully conceived remains a partial and tentative idea, and it is only because we know the system so well, we can immediately identify it when we see it half-formed. This constitute a pitfall for the student of mathematical history.

With the positional system in place it is natural to implement various algorithms of computation which we are painfully taught at school. (However, when it comes to making mental calculations, special tricks are usually more effective, than the slavish method of learned algorithms). The positional system took some time to diffuse and Europeans learned it through the Arabs, this is why we talk about Arabic numerals, although the numerals that the Arabs use look very different[8]. The positional system, such a boost to computation and thus to trade, was not established in Europe until late Medieval time. Some people claim that this simple invention is the basis for all technological development. However, one should not forget that computational devices, such as the abacus, have a long history. The abacus is a mechanical device which very efficiently represents numbers and allows computations on them, which were fully adequate for the purposes. In fact a skilled abacist is superior to a modern man with a calculator. In fact the positional system was an adaptation of those mechanical ones for the use of pen and paper. And forget not that the abacus is very close to a modern computer using pebbles rather than electronic circuits. Also the manipulation of an abacus is a physical fact independant of humans, except as when it comes to interpreting the outcome[9]. It is quite likely that the positional system developed as a way of transfering a mechanical device to paper, what had been automatic in the device needed to be marked in writing, hence the invention of the 'zero' as a simple place holder.

If there is a Civilization there is invariably some elementary mathematics based on practical needs such a trade and taxation. More interestingly though there is a need for astronomical observations to keep a calendar. This leads to much more interesting mathematics. Thus there is rudimentary mathematics to be found in Egypt and Babylonia whose importance and influence should not be underestimated. Both Civilizations survived for thousands of years, something we cannot take for granted that our Modern one will, and in particular Babylonian astronomy left behind very long series of data, testifying to a defining aspect of Civilization namely interaction over stretches of time greatly exceeding that of the life time of a man. Mathematics also developed in Persia, India and China. But as noted before none of them comparable in sophistication with the Greek. The development of a vibrant and sophisticated scientific tradition, including a mathematical, is not predictable, and thus nothing to take for granted, once the tradition is broken and fallen in disuse, it may very well not emerge again.

One pertinent example is the Arabic Civilization. As noted after a period of dramatic expansion there was a time for consolidation. Then there appeared in the eight century or so a cultural center in Baghdad under the Abbasid Caliph which soon acquired a reputation as a city of learning as represented by specifically the House of Wisdom an academic institution devoted to the humanities as well as the natural sciences in addition to mathematics and replete with the largest library of the world at the time. Thus Baghdad became the true successor of Alexandria. This was only possible through an atmosphere of tolerance and curisoity (the two tend to go together) that characterized Islam during its heyday. Thus the conditions for scientific inquiry was much more favorable than in Christian Europe at the time[10]. Although Baghdad was the unquestionable center, Arab culture spread all over its realm, in particular to Spain and Sicily. Gradually, however, the dedication to scientific inquiry faded away as religion became more and more important. It did not help that the Mongols sacked Baghdad in 1258 from which it never really recovered (just as Rome never really recovered from being (repeatedly) sacked by the Barbarians some eight hundred years earlier), but by that time the Golden Age was already long past. Thus no scientific revolution did take place in the Islamic world. This is not meant to imply that the Arabs were inferior to the Europeans, or that Islam was inferior to Christianity, only that the preservation of a tradition is a fragile matter and can as well go one way as another[11]. At the time when Arab civilization was at its zenith, it was superior to its western rival. It involved a preservation of the Greek heritage through the copying and translation of its texts. Just as it also incorporated the ancient cultures of the Middle East as well as the eastern one from India. It was not content with preservation it also built on it and developed it further and made some progress of a solid, if not revolutionary nature. It did improve and extend spherical trigonometry, and it is to the Arabs we owe the introduction of sine. One particular problem which came up in connection with religion was that of the *qibla*. It concerned the direction of Mecca to which every devoted Muslim should face in prayer. It presupposed that the Earth was a sphere and that latitude and longitudes of relevant locations were known. The first was learned from the Greeks, and should remind us once again that the roundness of the Earth was not something shown by Columbus, as many popular conceptions seem to indicate, but was established knowledge among the elite[12]. Furthermore while the determination of latitude is fairly straightforward, that of longitude is not, but requires an independent time-keeping piece that can be moved from one location to another[13].



Given the latitude (μ) and longitude (δ) of Mecca, and the corresponding data (λ, γ) for the location, we will know two sides of the relevant spherical triangle $(90 - \lambda, 90 - \mu)$ and the intermediate angle $(\delta - \gamma)$. From this the desired angle at P will be known. In order to actually compute it one may extend the arc to the equator O and then we have three right-angled triangles to play around with. The details are left to the reader[14].

There were some efforts to introduce algebra, the word itself is Arabic as most of you know, although no really convenient terminology was invented, which is very important to algebra which is based on almost mechanical manipulations. Some simple combinatorial problems were considered and solved, in particular the number of permutations of a given number of objects, as well as what we now call binomial coefficients. And in addition to sums of consequetive integers, formulas or rather methods for corresponding sums of squares and cubes were discovered, although they were hampered by the lack of good algebraic terminology. The interesting thing is of course that they were written down and documented, the results as such are not very remarkable, and well within the reach of any competent mathematician at the time, and hence rediscovered in many different mathematical cultures. They also adopted the positional system, which may have been the most important thing they transmitted to the West which was not already of that tradition. One should never forget that individuals were as clever and hardworking then as they may be now, and within the limited understanding and technology they were nevertheless able to achieve remarkable things. One of the Arabic mathematicians by name of al-Kashi solved the relevant cubic equation $4x^3 - 3x = t$ recursively in order to get a good approximation of the sine for 1^o knowing it (as a quadratic expression) for 3^o.

The European Dawn

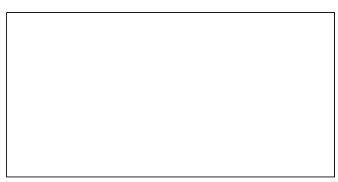
During the Dark Ages intellectual activity was confined to monasteries. It is an old institution which arises in many cultures and is usually associated with religions. It allows a small segment of the population to live in seclusion sheltered from the ordinary bustle of life, be it war, commerce or agricultural toil. The monastic life thus involves a lot of leisure be it disciplined. Much of that free time is taken up by religious observance for the good of the soul, be it of others or the practioners. When there is study, the study is usually confined to that of religious texts as in modern Muslim madrasas[15]. Yet, among the hundreds of thousands of monks and nuns, during a score of generations, there are a few who may be doing some useful work as far as to attract the attention of posterity. When it comes to mathematical work, there really is none beyond that of copying manuscripts. If I am to mention a single name, it is that of the English chronicler Bede of the 7th and 8th century. A prolific monk indeed, who wrote some sixty books, but is mainly remembered for his promulgation of our present convention of dating - the *anno domini*, which actually was invented a century or so before Bede by the Scythian monk Dionysius Exiguus. How accurate is that dating really? Dates cannot be added but subtracted. The dating system is only defined up to an additive constant, like a potential, but the difference between dates is independent of the dating convention. Thus given two dated historical events from the Dark Ages you can ask the objective question whether the difference actually correctly corresponds to the time elapsed between them. Could there be missing years as well as added ones? The calibration depends on historical records using different conventions. Only one thing can we be sure of, namely that after the general acceptance of the dating, every year since then is kept track of.

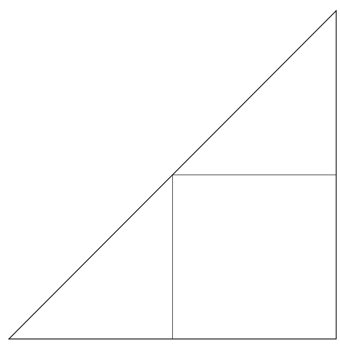
The end of the Dark Ages is usually associated with the Renaissance, and there is often a precise date - 1453, associated with it. Namely the date when Constantinople fell to Turkish invaders, and the Eastern Roman Empire finally collapsed, leading to a diaspora of monks and classical documents which reached Europe. In reality the process was much slower and lasted for a few centuries. One may mark its beginning with the first Crusades who took place in the 11th century to be followed with about half a dozen subsequent ones during a period of about two hundred years. The Muslim advance had been stopped by Martell (the grandfather of Charlemagne) during the early 8th century as already noted, but in the East the Byzantine empire was under pressure of Turkish invaders who had conquered most of the Arab Empire including the Holy land. The Crusaders wanted to reclaim the Holy Land and had initial successes which were not sustained. It was a sequence of wars of conquest and pillage and sullied by atrocities, also directed against fellow Christians, that shocked contemporaries. Nevertheless it led to an upswing in trade and commerce and contacts with a foreign non-Christian culture that also acted as a conduit for the knowledge of the Classical World. Because the Arabs after their initial frenzy came to copy and translate and hence Greek manuscripts, saving many of them from destruction and obliteration, thus to some extent making up for past misdeeds. In this way the Europeans came to encounter Greek culture through the detour of the Arabs. The European intellectuals fell in love with Greece culture, and this celebration would survive until today.

As we have noted monastaries were religious institutions in which intellectual work could be conducted. They led to general institutes of learning and universities appeared in Europe during the 13th century, notable examples being Oxford and Cambridge in England, Bologna in Italy and Paris in France. The syllabus consisted of the trivium and quadrivium, the first part was the elementary part consisting of grammar, logic and rhetoric, while the quadrivium consisted of geometry, arithmetic, astronomy and music. Those seven arts were the backbone of the liberal education. One should note that the quadrivium was very mathematical, in particular music was considered mathematical since the time of the Pythagoreansm who had established the basic principles between harmonious notes and integers. However, the mathematical education was very limited. Only parts of Euclid were studied, and then mostly by rote. Astronomy became very intertwined with astrology, and for many centuries astrologers were usually referred to as mathematicians, probably because of their extensive calculations. Astrology was initially frowned upon by the church, but in late medieval times it was very widespread and played a very important role in medieval universities (I refrain to speculate upon what plays that role in modern universities). In particular astrology was important in medicine!

The Catholic church which incidentally had been hostile to the Greek heritage tended to embrace it in late medieval times. Aristotle became their philosopher of choice, although as the philosopher Bertrand Russell has remarked, Plato would have been a more congenial choice. This was a result of an interesting development in which human reason started to occupy the same high ground as revelation. It was not enough to be told by authorities you also had to fathom it by your reasoning. I am thinking of the scholastics, with Thomas Acquinas (1225-74) as the leading light. The subjects pondered were on a very high level of abstraction far above the level directed at the 'vulgatea' with its idoltorous devotion to the Madonna, which would have been anathema to the Muslims. The church was the only outlet for intellectual activity available. And scholastic theology provided an opportunity for philosophy and the efforts to conciliate it with an elevated conception of God presented a challenge. From this period stems various attempts to prove the existence of God, the most elegant no doubt being the ontological one by Anselm. The argument goes as follows. 'God is the perfect being. To the attributes of perfection belongs existence'[16]. Scholasticism does not have very much to do with mathematics directly, but its tenor would reappear in mathematics with the introduction of Set theory and the Foundations of mathematics, some six centuries later. After all it is a question of pure thought taking command.

Some mathematical activity did also take place in Europe during the late Medieval period. But nothing spectacular but rather pedestrian. In retrospect the most intriguing mathematician was the French scholastic bishop Nicole Oresme (1320/25-82).





He had the idea of representing velocity for different times with a vertical segment proportional to the velocity, thus in the case of uniform velocity we get a rectangle and the distance covered is given by its area. This can be generalized. In the case of a velocity increasing uniformly with time, we instead get a triangle (or more generally a trapezoid if we do not start at velocity zero) and the distance covered is given by the mean velocity, meaning the average of the initial and final. In particular if starting from velocity zero, one quarter of the distance is covered during the first half and three quarters during the second. Thus we see that Oresme did anticipate some of the calculus later to be developed by Newton and Leibniz, but so did Archimedes and also som other mathematicians from different traditions. It is one thing to discover the rudiments of an idea retrospectively, quite another thing to develop it at the time as to have it make an actual influence.

Perhaps the most well known mathematician of the European Middle Age is Fibonacci. He is known for his compendium *Liber Abaci* which is basically a manual in how to do arithmentic including handling fractions, as well considering square and cube roots. His most lasting contribution was introducing so called Arabic numerals, i.e. the positional syste to Italian and hence Western traders. He also considered various mathematical problems of a recreational bent, such as computing the number of rabbits a couple of such can progressively give rise to. This led to the sequence $1, 1, 2, 3, 5, 8, 13 \ldots$ carrying his name.

All in all the contributions to mathematics in Europe during the medieval times was very modest, and in particular there were no systematic explorations.

Notes

[1]The Academy of Plato did survive, at least in a formal sense for many centuries A.D. until it was closed down by the East-Roman authorities.

[2]'His City of God' was his foremost theological text, while his 'Confessions' set an example for autobiography still very much alive.

[3]A variation of the proof that there are truths can be traced back to Augustinus. If there are no truths, this statement would be a true statement. But it leads to a contradiction, hence there must be truths. Note that the proof gives no indication of what could be a truth, except of course the self-referential statement that there is at least one truth, and that that statement is an example of it.

[4]One may compare Nietzsche, a devoted student of Classical Greek if any, as testified among other things by his breakthrough - Die Geburt der Tragödie (The Birth of Tragedy) - attacked the Christian compassion for the weak.

[5]And the sentiments have curried favor much later, so David Hume the liberal Scottish philosopher of the 18th century could express an admonishment about books. 'Does it refer to facts and figures? If not consign it to conflagration, as it can contain nothing worthwhile', or words to that effect.

[6]When it comes to the game of memory Chimpanzees reportedly perform better than humans. More remarkable though is their ability to understand that it is a game and be willing to play it.

[7] More so in the past before the advent of calculators.

[8] Still in India there are many dialects of numerals, the local buses in Mumbai as an example have a different set of numerals, and although the task is essentially trivial, it takes quite an effort to get used to it, and familiarity with it will never rival that of the well-known ones. This is a timely reminder that much of what we tend to take for granted is the effect of a previous effort.

[9] Thus the question of who the Romans could have calculated with their cumbersome system is moot to my mind. When computations were called for it was no doubt done by slaves using abacuses.

[10] The Muslim attitude towards Jews have been much more tolerant than the Christians. In fact Christianity was untouched on the Iberian peninsula and until our day large Christian communities have lived on in the Middle East, one may in particular think of the Coptics, predominantly in Egypt where they still constitute a large minority, but there are also Christian communities in Syria and Iraq. This benevolent attitude has changed however in recent decades, the reasons for that are not that hard to come up with.

[11] There is of course no dearth of explanations when it comes to illuminating historical facts such as these. Books have been written on the subject. Tempting, not to say exciting as it is to engage in such speculations, one should be wary of it.

[12] As to the man tilling the soil it is another matter. To most Modern people the spherical form of the Earth is only apparent through (putative) pictures taken from space. But merely sixty years ago such sensuous evidence was not available.

[13] Admittedly short-lived astronomical phenomena visible at two different locations would give you a means of computing time-difference by the relative positions of stars visavi the horizon.

[14] Once we find out the length of PM we can use the sine theorem for spherical triangles, stated and proved by the Arabs, to find out the angle at P. Using the sine theorem for $P'PO$ and $M'MO$ we find that $\sin PO : \sin MO = \sin PP' : \sin MM'$. We can also use that the two angles at P are supplementary.

[15] The term was originally refering to any educational institution, be it religious or secular, and that meaning still survives in Modern India, although in the West it is usually thought of as a school of religious indoctrination, even as a plant for terrorism.

[16] There are other proofs, necessarily built on the same type of abstract principles. One is that there has to be a first mover, that one cannot imagine an infinite regression, there has to be a first step. One may compare with the modern Big Bang theory, one of whose proponents being a Roman-Catholic priest George Lemaître (1894-66). It is clearly very congenial to Catholic theology.