Fifth Lecture

Renaissance Mathematics

Why was there such an upswing during the 15th century? An upswing usually referred to as the Renaissance. One date that is often quoted is that of 1453 when the Turks conquered Constantinople. During the Crusades Western Europe had come in contact with the East Roman Empire, known as the Byzantine, and the contacts had not always been benign. Constantiople was cruelly sacked by crusaders in X, but with the advent of the Turkish invasion, there were common military interests with the Byzantines. Acquaintance with Greek culture were much older, but during the Medieval times this was mostly relayed through Arabic, with the expulsion of Greek scholars, there was an exodus to the West bringing with them a lot of manuscripts, enabling a more direct contact with Greek civilization. But of course the ground was already been laid.

During late Medieval times Italy was very active. Internally through a lot of civil strife, the peninsula divided into many city-states (reminiscent of classical Greece), externally there was a lot of trade, the peninsula having a strategic position. This generated wealth and leisure and an upswing in the arts. There was a great demand for artisans, artists who could not only paint pictures but also plan fortifications and construct buildings. There was a lot of demand for engineering. This required practical knowledge of the kind not available in ancient documents. It also was pragmatic, what worked and how, not so much why, was the overriding question. This paved way for a more empirical approach. On the other hand theory was important, nothing as practical as a good theory, and here mathematics came into its own. Plato was rediscovered, and to Plato mathematics played a more important role than it did for Aristotle. In a way God became identified with mathematics. God created nature through mathematics, and by studying mathematics one was extolling God. It also made the study of nature more quantitative, and here we may see if we want a return to Pythagoras. Religion and mathematics in a sense fused. Mathematics was seen as the secure knowledge, something that the axiomatic presentation of Euclid certainly contributed to. In this way one could escape the strictures of dogma, while still paying lipservice to it. The Renaissance was a forerunner of the Enlightenment, or perhaps rather the Enlightenment was a logical consequence of the Renaissance as the latter gained momentum.

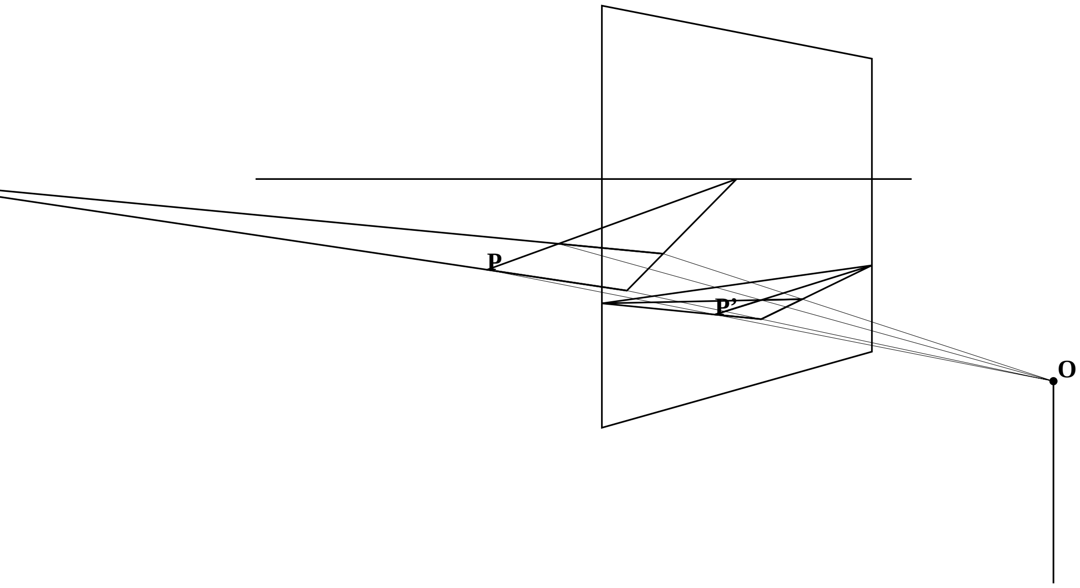
Concomitant with this study of nature there was a desire not only to understand nature, but to control it for the purposes of the well-being of man. In other words rather than to submit to the whims of nature man should dominate it. The man more responsible for that view than anyone else is Francis Bacon, and his vision very much permeates modern science. In particular he proposed that useful knowledge could be produced more or less automatically on an industrial scale provided one only knew how to read the book of nature properly. In other words everyone could be taught to observe and draw the right conclusions. This is a view which is very much in accordance with the popular view of science, to say nothing of the view shared by politicians. Science is a method that requires of its practitioners scrupulous objectivity and adherence to impeccable scientific standards, whatever is meant by those. It also reflects an almost unbounded optimism, to some extent serving as an inspiration to this day, that with the scientific attitude, no problem will remain insoluble and that it will continue to serve the wellbeing of mankind. In other words science is like a tap, you just turn it and water will flow to your content.

The Artist as Scientist

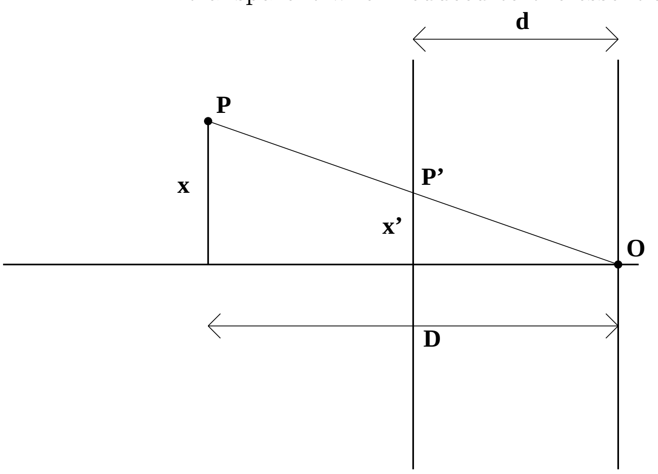
As noted the Artist was expected to be a jack of all trades, not only adept at painting and drawing, but also active as an engineer, an inventor and constructor, an architect, and what not. Thus Leonardo da Vinci was in no way exceptional as a phenomenon, only that he was so much superior to anyone else in his multi-disciplinary ambition. The age of specialization was to come later. Nowadays our view of artists is almost antipodal to that of scientists. While the latter are expected to be objective and to devote themselves to the practical problems of mankind, the artist stands apart, more dedicated to expressing his own subjective feelings than to address the material welfare of his fellow man.

An overriding concern was the study of nature, in particular its faithful depiction on a flat canvas of a 3-dimensional reality. The later was almost a purely mathematical problem, resulting in the so called laws of perspective. That problem was solved during the 14th and 15th century and needless to say da Vinci mastered it. But that was not enough to solve the problem of so called mimesis, to properly paint a human body you have to 'understand' it. It is not enough to make superficial observations, you need to acquire anatomical knowledge. Da Vinci did, and not only that, to paint a tree you need to understand how it is built up, and da Vinci observed that when a tree forks the sum of the surface areas of the sections of the branches equal that of the section of the original stem. Da Vinci made drawings, and a drawing differ essentially from a photograph, as a drawing is a report on an observational inquiry, not a mechanical reproduction, it involves not just registration but active interpretation. Thus the salient features are emphasized.

The principle of perspective is very simple. We are given a point O (the eye of the artist) and a plane Π (the canvas) and to every point P we draw the line OP which will intersect the plane in a point P', thus we get a correspondence between points in space and points in the plane. The image of a line L will be the intersection of the planes LO and Π, and given two parallel lines L, L' they will give rise to two non-parallel planes $LO, L'O$ (as they have at least the point O in common) which will intersect in a line l which typically will intersect Π in a point p, thus on the image parallel lines meet. Now consider the flat ground which typically is a plane F perpendicular to the canvas. An interesting plane H will be the plane parallel to F and passing through O. It will meet Π in a line h which will be denoted the horizon. It disconnects the canvas in two parts. Points on F will be mapped below the horizon, and more generally any point below H i.e. of height above F less than O. Points above H will then be mapped 'against the sky'. The further away a point on F is the closer it will be mapped to the horizon. No point on the horizon will, however, correspond to any point on F. One says that they correspond to points infinitely far away. Here we have the embryon of projective geometry which developed out of perspectives.

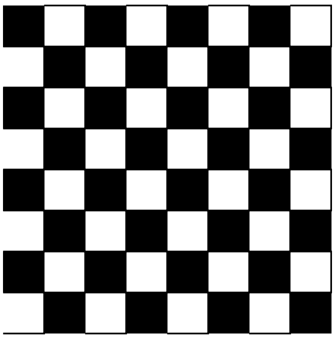


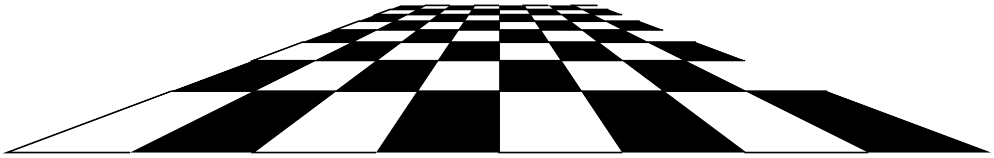
Now in modern algebraic representation in terms of coordinates, the whole thing can be easily described and explained. If O is given at (a, b, h) and Π by say $y = c$ and P by (x, y, z) it is straightforward to find the line OP and compute the image of (x, y, z) as the intersection of the line with Π. The principle becomes transparent when reduced to the essentials as in the figure below.

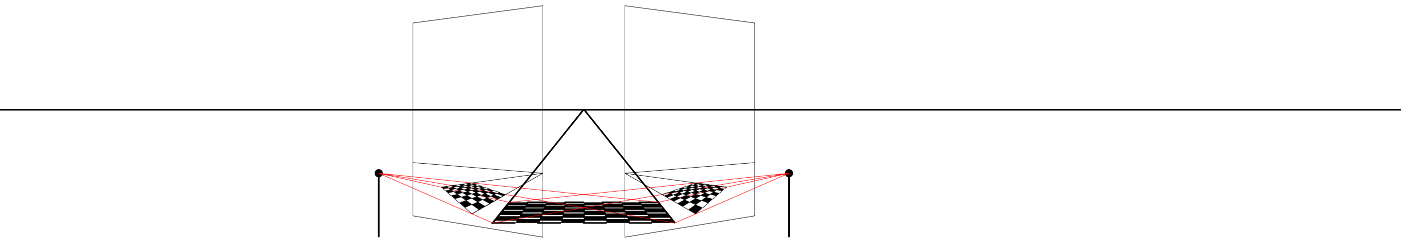


Note that $x' = \frac{d}{D}x$. Thus when D increases with x fixed we have that x' decreases. Things farther away looks smaller. If the eye is far away from the canvas but the objects are not (i.e. d is large compared to $D - d$) there will be a very modest foreshortening effect and the picture will look rather flat (the tele lens effect). On the other hand if d is small, the effects will be exaggerated (the wide-angle effect).

So let us look at a checkered board.







Celestial predictions

The prediction of the movement of celestial bodies has been a common theme through human civilization. This is one reason why spherical geometry was conceived and spherical trigonometry developed before plane trigonometry. As noted, spherical geometry is more basic than the plane Euclidean, as this is the one we have an intimate relation to through our visual sphere. It had many important applications, such as navigation and keeping calenders. Also, even more so, it had momentous applications to the fates of humans, a science known as astrology, and which was predominant during the late Middle Ages. (It went against the creed of Christianity, but that does not seem to have hampered its popularity. It touches on deep needs of the human psyche, which now seem to be supplanted by genetic determination.)

The most accurate preditions were provided by Ptolemy's system of epicycles. Its basic assumptions were that celestial bodies moved in circles with constant angular velocity. But one circular movement was not enough to predict the movement of a body, you needed many circular movements superimposed on each other. In the end Ptolemy presented an elaborate system of close to eighty circles. He was well aware that his stratagem was, what we now would call a mathematical model, a scheme intended for calculation, not necessarily with any literal ontological interpretation. In order to adjust his scheme to observation he showed great ingenuity. By not compromising on his basic assumptions, he showed great flexibility as to how to center the various epi-circles. His approach is very modern, most of so called scientific fine-tuning of mathematical models consist in such tinkering. When it comes to predicting positions on the celestial sphere of bodies, it is fully adequate, higher and higher precision is achieved by adding more and moreepi-cycles. One should also be impressed by the consummate skills that went into such models, involving lot of calculations and visual imagination.

Copernicus simplified Ptolemy by putting the sun in the center with an earth revolving around its axis as well as around the sun. As a mathematical model there was little to object to, on the contrary it gave as accurate predictions and in a simpler way to boot. What was problematic was the ontology of it. It had striking counter-intuitive consequences. If the Earth was rotating at such speed, how come we were not thrown off? And if the Earth rotated around the sun, how come we noted no parallax? Francis Bacon rejected the hypothesis as absurd in obvious contradiction to the senses. And mind you the world of the senses is the world of which we have the most intimate connection, while theories are just born out of our imagination. This is called an instrumental point of view and had its early proponents in the Middle Ages, such as Occam with his proverbial razor, and is still very much in vogue, especially among the so called Post-Modernists. Thus the Copernican point of view had consequences well beyond the technical problem of positional prediction.

Mathematical Advances

The Renaissance involved foremost a discovery of nature and thus an extroverted 'Weltanschauung' in repudation of the Classical Christian one of salvation in a next world. It created an important niche for the role of mathematics, but by itself it did not immediately involve any spectacular development of mathematics as a science. One, however, merits special attention, and that is the solution of the third and fourth degree equation concomitant with the advancement of algebraic notation.

Cardano and the solution of the third and fourth degree equations

Tartaglia, Cardano, Ferrari and Bombelli are the main names connected with the explicit formulas for finding the roots. Of those Cardano was the most colorful and flamboyant character and the one making the deepest impression on posterity. The so called Cardanos formulas was supposedly given to him by Tartaglia under oath of secrecy, but Cardano decieved him and had them published under his name anyway. The procedure may be thought of morally reprehensible but as is not unusual in those cases, highly successful. To understand what is going on let us consider the quadratic equation known to the Babylonians. The key to its solution is the completion of the square, this is an idea that immediately gives the solution, and it is important to realize that this is not the same thing as giving a formula for the solution. A formula is indeed a neat thing, as we will see, but not necessary. The Babylonians had no formula for the solution only a strategy. The strategy can be clothed in words and be applied to each particular case. A formula, rightly understood, is just an encoding of the strategy, and to understand it, it is not enough to learn it mechanically but to learn the strategy behind. From a modern point of view what is important about an equation is not its solutions per se, but the linear space generated by its solution. If the equation is irreducible the dimension of this linear space is given by the degree of the equation. This linear space defines a field, and every equation thus defines a unique field, but conversely every field defines many equations, and the point is to find some kind of canonical one. In the case of quadratic equations, a natural equation is of the type $x^2 = d$. Given a general equation $x^2 + ax + b$ we can write it as $(x + \frac{a}{2})^2 + b - \frac{a^2}{4} = 0$ and setting $y = x + \frac{a}{2}$ we reduce it to $y^2 = D$ where $D = \frac{a^2-4b}{4}$. Another way of looking at it is to consider the trace of elements in the field. The trace is a linear function and its value on a constant k is $2k$ and in general it is given by $-a$ if x satifies $x^2 + ax + b = 0$. Thus if the trace is $-a$ of x it will be 0 for $y = x + \frac{a}{2}$ and thus y will satisfy an equation of type $y^2 = D$. Now for those equations we give the solutions $y = \pm\sqrt{D}$ but this does not in any way make any real progress, it is just a short form of saying $y^2 = D$ (where $\sqrt{D}$ is the positive root). In order to get a numerical value we need successive approximations, something which had been known for a long time. Now that the trace is in fact a linear function is far from trivial. If $x_i^2 + a_i x_i + b_i = 0$ are given for $i = 1, 2$ it is not so easy to show that $x_3 = x_1 + x_2$ satifies an equation of type $x_3^2 + (a_1 + a_2)x_3 + b = 0$ for some b. A modern argument would go something like this. Consider the linear space of all numbers $\eta = \lambda + \mu x$ where x is the solution of some quadratic equation. By construction it is closed under addition, but also because of the special nature of x it is also closed under multiplication. In particular the map $\eta \mapsto x\eta$ is a linear map and can be represented by a two by two matrix. Two such a matrix we can associate the trace which depends linearly on the matrix. Incidentally the characteristic equation of the matrix is the same as the qudratic equation of x. Such an approach would not be available until the 19th century although it is quite easy and elementary, testifying that even more important than good terminology are good concepts. We should also note that the expression $4a - b^2$

which turns up is very important. It is the discriminant of the equation and it is equal to zero iff we have a double root, i.e. the quadratic polynomial is a square. If it is positive then we have two real roots, and if negative two complex conjugate roots. If it is a square it means that the solution to the equation is to be found in the given field itself and that the numbers $\lambda + \mu x$ is actually one-dimensional, as the x can be identified with a constant. Thus from the modern point of view the field of the coefficients of the equation is crucial, and the field being the smallest field over the rationals containing them. Such distinctions would not to be clear to the ancients, the notion of a field being unknown to them. As Gauss noted 'notions' are more important than 'notations'.

What the ancients wanted to do was to split up the solution of higher order equations by reducing them to simpler ones like $x^n = d$ and thus to give formulas involving higher roots. As to a cubic equation it could be reduced to the form $x^3 + px + q = 0$ although for the Old Italians they were uneasy about negative numbers so they considered many subcases such as $x^3 + px = q, x^3 + q = px$ where the coefficients were tacitly understood to be numbers i.e. positive. This we would consider as a psychological hang-up and force them to repeat essentially identical arguments unnecessarily. Now, the trick of Cardano is to consider x as a sum $x = u + v$ we then get $(u + v)^3 = u^3 + v^3 + 3uv(u + v)$ now set $3uv = -p$ and try to find u, v such that $u^3 + v^3 = -q$, if so we get the identity $x^3 = -px - q$ i.e. $x^3 + px + q = 0$[1]. This leads to the problem of finding two numbers whose sum and product are known which reduces to a quadratic equation known since classical times. The reason for this is of course the connection between the roots of a quadratic and its coefficients, clear through $(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$ which was not explicitly written down at the time. Now we have the problem of finding u^3, v^3 knowing its sum $-q$ and its product $-\frac{p^3}{27}$. We thus have to find the solutions of the quadratic equation $x^2 + qx - \frac{p^3}{27} = 0$. We can write down its solutions by the quadratic formula (which you can always rediscover every time you complete a square) giving $u^3 = -\frac{q}{2} + \sqrt{\frac{27q^2+4p^3}{108}}$ and $v^3 = -\frac{q}{2} - \sqrt{\frac{27q^2+4p^3}{108}}$. Now you take the cube roots of each expression. To do so is only determined upon a cuberoot of 1 of which there are two in addition to the obvious. And those factors cannot be chosen arbitrarily as the product uv is fixed. Those things were not perfectly clear to them, after all the two other cube roots are non real numbers. Anyway Cardano came up with the formula

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{27q^2 + 4p^3}{108}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{27q^2 + 4p^3}{108}}}$$

or as he would have put it prefering a numerical example (say $p = -6, q = -40$ leading to $x = \sqrt[3]{20 + \sqrt{392}} + \sqrt[3]{20 - \sqrt{392}}$)

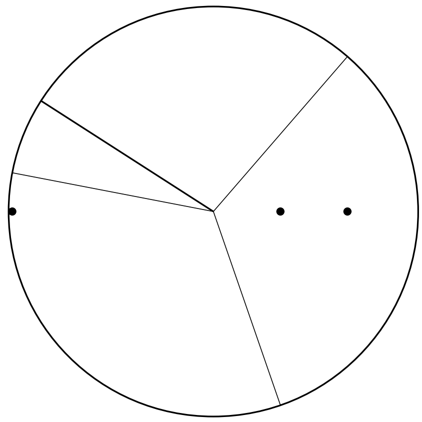
$$\Re v : cu.20p : \Re 392p : \Re v : cu20m : \Re 392$$

where $\Re$ denotes square root of, while v means that everything that comes after should be under the domination of the symbol $\Re$. The cu means that we should take the cube root, while p and m stands for '+' and '-' respectively. Now the importat thing to remember is not the formula itself but the procedure. A formula is easy to forget or remember wrongly, but with a procedure this is different. However, the formula presents some strange surprises. Crucial to the formula is the quantity $D = (\frac{q}{2})^2 + (\frac{p}{3})^2$. This can very well be negative, but then what will be the meaning of its square root, and how do you take cuberoots of the $-\frac{q}{2} \pm \sqrt{D}$?

Let us make a short digression on $x^3 + px + q$. The curve $y = x^3 + px$ is a cubic with a local maximum and minimum provided that $3x^2 + p = 0$ i.e. when $p < 0$ the values will be given by $y = (x^2 + p)x = -\frac{2}{3}px$. If the value $-q$ lies between those two values, there will be three real roots, otherwise just one. This is expressed by $q^2 < \frac{4}{9}p^2x^2$ and as $x^2 = -\frac{1}{3}p$ we get the condition $D = (\frac{q}{2})^2 + (\frac{p}{3})^3 < 0$. I.e. when we have three real roots the formula does not make sense, but if manipulated in some formal way, it will in the end give the right result. This is truly a mystery the type of which often will occur in the history of mathematics. So the formula is very awkward when applied to the case of three roots, and how do you get all three roots from the formula? The trick is of course to look at all three roots of $x^3 = d$ meaning prefixing them with a cuberoot of 1 chosing the inverse (or conjugate for the second).

Let us look at a particular example where we already know the roots, say $(x-1)(x-2)(x+3) = 0$ i.e. $x^3-7x+6 = 0$ we get the discriminant $D = -\frac{100}{27}$ and thus we will have to look at cube roots of $-3+\frac{10}{3}\frac{i}{\sqrt{3}}$ and $-3-\frac{10}{3}\frac{i}{\sqrt{3}}$ respectively. Those being complex conjugate the sum will be real (also if prefigured with complex conjugate cube roots of one), but how to compute the actual values? How do we take cube roots in practice of complex numbers? We put them into polar forms $re^{i\theta}$ and consider $\sqrt[3]{r}e^{i\psi}$ where $\psi = \frac{\theta}{3} + \frac{2\pi n}{3}$ for $n = 0, 1, 2$. Note that $\frac{2\pi n}{3}$ are the cube roots of one. Thus we can as well consider Vieta's trigonometric solution of the cubic.

This hinges on the identity $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$ obtained by iteration of the addition formulas for cosine. We are thus considering the equations $4x^3 - 3x = t$ for $|t| \leq 1$ (as t takes values of cosine). Multiply it with $2\lambda^3$ and we can write $(2\lambda x)^3 - 3\lambda^2(2\lambda x) = 2t\lambda^3$, or if we set $y = 2\lambda x$ simply $y^3 - 3\lambda^2 y = 2t\lambda^3$. If we have a general equation of the form $y^3 + py + q = 0$ we see that $p < 0$ and we should choose λ such that $\lambda^2 = -\frac{1}{3}p$ then $q = -2t\lambda^3 = \frac{2}{3}pt\lambda$. In order for $|t| \leq 1$ i.e. $t^2 \leq 1$ we need $q^2 = \frac{4}{9}p^2t^2\lambda^2 = -\frac{4}{27}p^3t^2 \leq -\frac{4}{27}p^3$ thus $D \leq 0$.



Returning to the cubic above we should choose $\lambda^2 = \frac{7}{3}$ and from $-6 = 2t\frac{7}{3}\sqrt{\frac{7}{3}}$ we should choose $t = -3\frac{3}{7}\sqrt{\frac{3}{7}}$. Set $t = \cos\theta$. The solutions will then be given by $2\sqrt{\frac{7}{3}}\cos(\frac{\theta+2\pi n}{3})$ for $n = 0, 1, 2$. An actual value for θ will then be $147.32\ldots^o$. In the figure to the left we see the three real solutions $1, 2$ and -3. Note however that the value of θ is transcendental and that we would never be able to get the exact solutions in this way. In fact the Cardano formulas turned out to be fairly useless when it came to find actual numerical values of solutions to cubics and other strategies would be developed. The great interest of those formulas were algebraic and were not revealed until the 19th century.

It may also be of some interest to present Vietas alternative solution to the cubic. Given the cubic in the standard form $y^3 + py + q = 0$ we make the substitution $y = z - \frac{p}{3z}$ transforming it into $z^3 - \frac{p^3}{27z^3} + q = 0$ which is a quadratic in z^3 given by $z^6 + qz^3 - \frac{p^3}{27} = 0$. Then solve it and we regain the Cardano formulas. The connection should be obvious. If $z = u$ then $-\frac{p}{3z} = v$ and both of them are solutions to the equation above.

From the modern point of view we have to cases. Either the quadratic equation $X^2 + qX - \frac{p^3}{27} = 0$ has solutions U, V in the field K generated by p, q, this happens exactly when the discriminant $q^2 + \frac{4p^3}{27} = 4D$ is a square in K. Or this is not the case and so the quadratic field L generated by K and $\sqrt{D}$ makes up a 2-dimensional vector space over K. There are many choices of basis, one is given by $1, \sqrt{D}$ another one by $1, U$ or even if you prefer U, V. In L we can find a cubic equation $z^3 = U$ and denote one solution by u. Then the field M generated by L and u is six-dimensional over K and a basis over L will be given by $1, u, u^2$ and over K by $1, U, u, uU, u^2, u^2U$ or if you prefer $1, u, u^2, u^3, u^4, u^5$. There is an automorphism τ of L over K given by $u \mapsto v(= -\frac{p}{3u})$ which extends to the whole of M. Now assume that there is a non-trivial cuberoot ρ of 1 in M. Two cases occurs, it could either have been present already in K or appearing in L. The discriminant of the quadric is given by $\frac{4p^3+27q^2}{27}$ which up to a square can be written as $\frac{1}{3}(4p^3 + 27q^2)$. In L the discriminant is made into a square. We have $\rho \in K$ iff $-\frac{1}{3}$ is a square in K and if so $-d = 4p^3 + 27q^2$ (the discriminant of the cubic) becomes a square in L. If ρ appears in L means that $-\frac{1}{3}$ is a square in L and hence also $-d$. If $-d$ is already a square in K then automatically $-\frac{1}{3}$ becomes a square in L and thus $\rho \in L$. Now we can define $\sigma(u) = \rho u$ and we get an automorphism of order three. If $\rho \in K$ already then σ and τ commute and generate $\mathbb{Z}_6$. If ρ appears first in L then $\tau(\rho) = \rho^2$ and thus $\tau\sigma(u) = \tau(\rho u) = \rho^2\tau(u) = \sigma^2\tau(u)$ and we get the group S_3. L is the fixed field of the cyclic group generated by σ which acts as the automorphisms of the field extension M over L. In the first case $\rho \in K$ we may also take the fixed field L' of the group generated by τ, this will be the cubic extension generated by the root x of x^3+px+q and M will be a quadratic extension of L'. Now u, v do not belong to it and hence will satisfy a quadratic equation over L' whose roots will be u, v as $\tau(u) = v$. Thus the trace and determinant will be given by $u + v = x$ and $uv = -\frac{p}{3}$ and thus they will satisfy the quadratic equation $Z^2 - xZ - \frac{p}{3} = 0$ which is easily checked. Furthermore $\sigma(u) + \sigma(v)(= \rho u + \rho^2 v)$ and $\sigma^2(u) + \sigma^2(v)(= \rho^2 u + \rho v)$ will both be invariant under τ and hence belong to L' which will then have an automorphism of order 3. In the other case those two roots will be permuted by τ and hence not belong to L' they will satisfy a quadratic equation over that field and their traces will be $-(u + v) = -x$ and their determinant $u^2 - uv + v^2 = (u + v)^2 - 3uv = x^2 + p$ and thus the quadric will be given by $Z^2 + xZ + (x^2 + p) = 0$. If we compare the two discriminants we will get $(\frac{4p}{3} + x^2)$ and $-(3x^2 + 4p)$ as they determine the same quadratic extension M of L' they have to differ by a square. In fact they differ by a factor -3 which is indeed a square in M. In this case multiplication by ρ will cyclically permute $L', \rho L'$ and $\rho^2 L'$ which are not cubic Galois extensions of K.

Let us look at the case $K = \mathbb{R}$. If $D > 0$ then there is a real solution $\sqrt[3]{d_1} + \sqrt[3]{d_2}$ (where d_1, d_2 are the roots of the associated quadraric, but the two other cube roots are not real. On the other hand if $D < 0$ then $L = \mathbb{C}$ and in particular $\rho \in L$. The roots are then $\sqrt[3]{\rho^i d_1} + \sqrt[3]{\rho^{2i}\overline{d_1}}$ for $i = 0, 1, 2$ and which are closed under conjugation and hence real.

Ferrari a young pupil of Cardano managed to solve the quartic (or biquadratic) equation by reducing it to a cubic. In modern notation we start out with

$$x^4 + px^2 + qx + r = 0$$

where we have gotten rid of the cubic term. Now rewrite it as

$$(x^2 + p)^2 = x^2 + 2px^2 + p^2 = px^2 - qx + p^2 - r$$

Now perturb p to $p + y$ and write

$$(x^2 + p + y)^2 = (p + 2y)x^2 - qx + (p^2 - r + 2py + y^2)$$

Now we want the right hand side to be a square, which means that the discriminant of the quadratic should be a square i.e.

$$4(p + 2y)(p^2 - r + 2py + y^2) = q^2$$

which amounts to solving a cubic in y. Once this is done we are reduced to solving the equation

$$A^2(x) = B^2(x)$$

where A, B are linear in x, which amounts to solving the two quadratic equations $A + B = 0$ and $A - B = 0$.

As an illustration let us look at the equation

$$x^4 + 8x - 7 = 0$$

This leads to the equation

$$(2y)^3 + 28(2y) - 64 = 0$$

By inspection one root is given by $y = 1$ the two others will be given by $y = \frac{-1\pm\sqrt{-31}}{2}$. Setting $y = 1$ we have

$$(x^2 + 1)^2 = 2(x - 2)^2$$

which leads to

$$(x^2 + 1 + \sqrt{2}(x - 2))(x^2 + 1 - \sqrt{2}(x - 2)) = 0$$

with the four roots $x = -\frac{\sqrt{2}}{2} \pm \sqrt{\frac{4\sqrt{2}-1}{2}}$ and $x = \frac{\sqrt{2}}{2} \pm \sqrt{-\frac{4\sqrt{2}+1}{2}}$ and

What happens if we instead choose another root say $y = \frac{1}{2}(-1 + \sqrt{-31})$? We cannot very well get other roots to the quartic, after all there are only four. But if we get the same answer, what happens to the mysterious $\sqrt{-31}$?

We obtain

$$\begin{aligned}(x^2 + \tfrac{-1+\sqrt{-31}}{2})^2 &= (-1 + \sqrt{-31})(x^2 - \tfrac{1}{4}(-1 - \sqrt{-31})x + \tfrac{(1+\sqrt{-31})^2}{64} \\ &= (-1 + \sqrt{-31})(x + \tfrac{1}{8}(1 + \sqrt{-31})^2\end{aligned}$$

We thus get the quadrics

$$x^2 + \frac{-1 + \sqrt{-31}}{2} = \pm\sqrt{-1 + \sqrt{-31}}(x + \frac{1}{8}(1 + \sqrt{-31})$$

Those will not be the same as above, but if we combine the roots in an other way say look at

$$-\frac{\sqrt{2}}{2} + \sqrt{\frac{4\sqrt{2}-1}{2}} + \frac{\sqrt{2}}{2} + \sqrt{-\frac{4\sqrt{2}+1}{2}}$$

which simplifies to

$$\sqrt{\frac{4\sqrt{2}-1}{2}} + \sqrt{-\frac{4\sqrt{2}+1}{2}}$$

square it and we will obtain

$$-1 + 2\sqrt{\frac{4\sqrt{2}-1}{2}}\sqrt{\frac{-4\sqrt{2}-1}{2}} = -1 + \sqrt{1 - 32} = -1 + \sqrt{-31}$$

which tallies with the above. That the product of the two roots will as well will be left to the curious reader.