Sixth Lecture

Mathematics and Science

The 17th century brought about a revolution in mathematics and natural science, the consequences which are still very much in effect today. In fact all the major scientific ideas can be traced back to the founding fathers of that century.

The revolution did not come out of a void but fitted in a tradition started and developed by the Greek. So in what way was it contigious with the Greek tradition and in what way did it differ?

First the two portal figures are Descartes and Galileo, who were followed by Newton and a whole hoist of minor but exceedingly brilliant individuals such as Huygens, Halley, Hooke etc. Descartes and Galileo took at their point of departure Euclid, i.e. the deductive rational method. They deviated perhaps not so much from the Greeks as they took exception to the way the Greek scientific tradition had been distorted during the Medieval ages.

Aristotle was the greatest scientist during the Classical world, and as to his versatility maybe the greatest ever, although it certainly helped at the time that so much of natural knowledge was virgin territory. The Greeks were very curious and observed nature keenly, among other things testified in their language and the metaphors they employed. Aristotle in particular was a keen observer. He was a student of Plato but differed from his advisor in that he put much less emphasis on mathematics and abstract ideas, and more on the sensual world around him, which he set out to classify and categorize, natural occupations for a pioneering explorer. It stands to reason that a single individual taking on such a wide variety of tasks will err at times. In the Greek tradition the teachings of Aristotle would have been submitted to criticism and modification, but as they entered the Western Canon, they became Holy Scripture. What Descartes and Galileo rebelled against was not Aristotle per se as much as the veneration he was accorded. Thus they resumed the Greek tradition of knowledge to be criticized and based not on authority but on a shared rationality. Ultimately it was the human intellect who should be the judge, and the intellect thrives on clear and distinct ideas. Thus Descartes as well as Euclid wanted knowledge to be securely based on indubitable principles. The vision of Descartes was wider than that of Euclid, he wanted to include not just physical space but general knowledge of nature. He was a scientist, and as such he was a philosopher and a mathematician. It is instructive to ponder his indubitable principles to which he had arrived after much soul-searching doubt.

i) His famous lines *cogito, ergo sum* represents the rock-bottom of his journey of doubt. In fact Descartes as Galileo wrote in the vernacular, so the original is in French (*Je pense, donc je suis*) but it was not taken seriously until it was translated into Latin. It represents the idealistic position that it is our thoughts with which we are most familiar and hence should take as our point of departure.

ii) Each phenomenon must have a cause.

iii) An effect cannot be greater than its cause.

iv) The mind has innate in in it the ideas of perfection, space, time and motion.

From this axioms he derived in true Scholastic spirit the existence of God. To modern minds this might be something of an embarrassment, and it very much continues to be one. Descartes is known for this split known as Cartesian Dualism. On one hand he is a materialist seeking in Nature mechanical explanations, guided by ii), on the other hand he is an idealist (as noted by i) and iv)) excepting the mind from the physical, mechanical world. How these two parallel worlds interact is a thorny technical philosophical question, which has, as already noted, led to embarrasment. In modern parlance, Descartes is both a scientist and a humanist.

The mechanical point of view has guided natural science ever since and provides its overriding paradigm. Descartes set out to methodically understand nature via this mechanical methodology. He put great store at a methodic approach and illustrated that in the appendices to his 'Discourse de la méthode' in which he not only treated Geometry but also Physics and Biology. In the latter he used experiments and rivaled Harvey in the discovery of the circulation of blood (although where they differed, Harvey turned out to have been right). However, only in mathematics have his discoveries turned out to be of lasting value. To Descartes mathematics was a tool to understand nature and he supposedly looked down on pure mathematics as an idle pursuit.

Galileo, although having done no mathematics comparable of influence and importance to Descartes, made the connection between mathematics and natural science even more explicit. Famous are his words that the language of Nature is mathematics and to understand nature you need to be a mathematician and think in terms of mathematical objects. Incidentally his claim that mathematics is a language is clearly a metaphorical statement, which, however, have been interpreted literally by modern educators and led to much silly confusion. The intimate connection between mathematics and the natural world, meaning essentially mechanics, celestial as well as terrestrial, would kick-start mathematics during the 17th century and be the dominant feature of 18th century mathematics. Only with the 19th century would the notion of pure mathematics arise and developoe in earnest and concomitant with it the greater need of an interest in rigor, which would lead to a crisis in the next.

To bring about this union one had to strip nature bare to the bones and disregard anything that was not basic and objective. That meant to concentrate wholly on primary characteristics of bodies, such as extent and weight, and disregard secondary such as color, and smell, which were subjective and transmitted by sensory organs to be distrusted. This distinction was already made by the ancients, and in the contemporary philosophy of Locke, it played a central role. By this redution one could concentrate on the measurable and hence amenable to quantitive analysis.

The great advance and the basic difference from Aristotle is not to look for ultimate causes of events but for immediate, not to ask why, but to show how. This brings out the difference between Nature and the Mind as it is generally understood, and Cartesian Dualism is still part of the mindset of most people. If you write about human history, the question of why is paramount, it is not enough just to show how, you have to understand something of the motivation. The same thing goes for criminal investigations, finding a motive goes a very long way to solving a crime. Thus the laws that were to be sought were quantitive and describtive laws which did not at all probe into the question of finding ultimate explanations. Those were laws which could be mathematically manipulated, and conversely laws which led to mathematical problems which necessitated the further development of mathematics. Such laws necessitated experiments, maybe not so much as to their general structure but to determine the constants involved. Those you could not figure by pure thought.

It is easy to say that Aristotle and the Greek only speculated and did not look at nature, while the scientific pioneers were more empirical. Galileo rarely resorted to experiment, if he did experiments it tended to be of the arm-chair type, namely so called thought experiment. The famous story of dropping stones from the conveniently leaning tower of Pisa is apocryphal. The idea that stones fell at the same rate independantly of mass in vaccum was due to a mental extrapolation. Galileo had not access to a vaccuum to drop stones into. It was a daring hypothesis that was mathematically elegant. This search for mathematical beauty and simplicity has been a leading and fruitful strategy ever since. On the other hand finding the constant rate of acceleration requires empirical testing. Such a testing only makes sense using the law, i.e. accepting, at least provisionally, a scientific theory. As Aristotle did not conceive of such laws, there was no compelling need for him to set up experiments. It is very hard, not to say impossible, to set up an experiment to test an ultimate cause or a teological explanation.

Newton carried this analysis even further. His postulating his law of gravitational attraction had striking mathematical consequences, such as the Keplerian laws in the case of two bodies, and the necessary pertubations when there were many bodies. (As you all know, the orbits of planets are not ellipsis, they do not necessarily close up, they can be approximated by ellipses, but those approximations will over time undergo sublte changes as to location and shape, the axi will change directions, the excentricties vary, often in a nearly periodic manner due to the pertubations caused by other planets.) His law, however, was strange, and did meet with severe criticism at the time, some of which Newton agreed with. The law gives no reason why such a law should exists nor why bodies exert gravitational attraction to each other. The discarded theory of Descartes was in that regard more satisfying. He abhorred the vaccuum, and gave a mechanical explanation in terms of vortexes. In the end Newton won out as being the most fruitful and the one more compatible with actual observations. Science does not explain the unknown so much by the known, as it explains the known by the unknown.

A conception of empirical science that adhers better to predominating views is that of Francis Bacon. To whom Nature was an open book to be read. Only do observations, unclouded by prejudice and preconceived notions, and you can read off knowledge. The point of science is to get power over Nature and thus to increase the well-being of mankind. Most people in the general Public, to say nothing about politicians, find this an accurate and desirable view of science. One should, however, keep in mind that Bacon did not believe in the rotation of the earth, neither around its own axis, nor around the sun, common sense contradicted such a notion. If science is to make revolutions, to go beyond to the next level, it does need to make daring hypothesises, contradicting common sense.

Galileo

Galileo Galilei was born on February 15, 1564. His father was a musician, an accomplished player of the lute, and had apparently discovered that the pitch of a string varied at the square root of the tension. A non-linear addition to the relationships between music (or tones) and mathematics discovered and presented by Pythagoras. This is an illustration that artists of the past were more of inquisitive scientists of today than expressionists of emotions, as well an indication that he grew up in an intellectual family. Although the father's ambition for his son may have been musical as were fulfilled by some of Galileo's siblings, economic difficulties for the family suggested that the son instead study medicine at the university of Pisa, but Galileo's chance encounter with mathematics made him beg to change his course of study and he became professor of mathematics at the same university at the age of 25 at 1589. The year thereafter he encountered the heliocentric system of Copernicus, which would have a crucial effect on him and almost a fatal one. One year later his father died and the following year he moved to the university of Padua where he stayed until 1610.

Already as a young medical student he had noticed that the chandeliers in his church swung at the same rate independently of their amplitudes, which varied due to gusts of winds inside and other disturbances. To check his hypothesis he used his pulse rate as a clock and at home experimented with different pendulums as to be able to directly make comparisons. It is to be noted that at the time there hardly were any time-pieces to measure short durations of time. This example illustrates many things. First a general inquisitive mind alerted to the kind of observations most people would not care about; then the forming of a hypothesis and the ingenuity of making a focused observation under controlled circumstances, a so called experiment. To set up such experiments require a lot of ingenuity and technical skill. The idea of using your pulse as a clock is one such, the other to make things even more precise by letting the pendulum become their own time-pieces and thus comparing them directly. How do we know that the periods are the same from one time to another? Philosophically we cannot tell, the question makes no sense really, as we cannot, unlike with a measure which can be moved freely in space, go back in time. Thus instead of pondering the metaphysical question, which admits of no solutions, he concentrated on what he could watch and do, in particular whether there were any consistency in the periods. This points to a very important point: Galileo was not interested in why-questions only how-questions describing what was going on, and most importantly the how-questions were vouched in quantitative ways, which permitted computations to be made and the results of which had meaning and could be fed back[1]. Later on Galileo would also turn his attention to falling bodies suspecting that velocities increase linearly with time, or in the language of the time, were proportional to time. That the distances covered hence we proportional to time squared then becomes an non-empirical fact derived from the first empirical. Galileo was not the first who realized this fact, this insight can also be gleaned from the writings of Oresme, but he was the first to do so systematically and to ground them firmly in experiment. And once again we meet with difficulties, falling bodies fall too quickly to be observed with any accuracy by human senses. Galileo cleverly simplified observations by using (slightly) inclining planes, no doubt varying the inclinations, and making many careful observations. But observations are not enough, you must also be able to go beyond them and to discern general principles which cannot always be empirically verified. The notion of inertia is one such. Contrary to what most people think, and in particular what the Greeks thought or at least what Aristotle claimed, movement does not need a constant force to be kept alive. Galileo postulated that a body in uniform motion will keep on moving at the same rate, unless there are obstructions. From a purely logical point of view the statement is tautological, if it does not there must be obstructions of some kind making their presence felt by falsifying the assumption. Such statements cannot be empirically verified, they are figments of our imagination, or rather mental fruits of the same, which turn out to have a very crucial effect in our 'understanding'. Galileo was a bit unsure of the principle and it was not clearly stated until Newton made it one of his fundamental laws, pointing out that uniform motions entails moving in straight lines (not circular ones, a possibility left open by Galileo). Any deviation from uniform rate along a straight line means that a force must have acted. Force generating a proportional acceleration known as Newton's second law [2] $F = ma$. Later on the meaning of those statements were queried. Did they say anything? What is force? Can this be used as the definition, and that forces are just virtual entities with no real existence only convenient fictions, as the called instrumentalists with Ernst Mach claimed at the end of the 19th century? And what is mass? Is that also defined by the equation?[3]. Now in a terrestrial setting you may make the principle believable by looking at objects moving with little friction and air-resistance, which is not so easy to bring about; in a celestial setting though, the principle makes sense of the movement of bodies, thus it was crucial for Newton. The inertial principle can also be codified in the notion that reference systems which are in uniform movement with respect to each other are equivalent. This is at least implicit in Galileo and have retrospectively been termed the Galilean principle and served as a basis, suitably modified, for Einstein's theory of special relativity. With Galileo objections to the rotation of the Earth to the effect that it would create strong winds could be laid at rest. As such it amounted to a definite advance over Greek Aristotelian physics and laid the foundation for modern Newtonian physics. As can be seen the actual accomplishments of Galileo may be thought of as slight in bulk compared to what others would contribute later, but they were fundamental and set the stage for future developments.

In particular Galileo recognized that the path of a cannon ball was that of a parabola. This was clear from the principle that one could decompose the movement in two components, a horizontal one of uniform motion, and a vertical one of a free fall with constant acceleration. Thus the horizontal movement was linear and the vertical was quadratic. The decomposition of a force into components (the so called parallelogram law) was explained already by Stevin in the early 16th century and was probably known to the Greeks especially to Archimedes. For us it is easy to see that this defines a parabola, at the time of Galileo it was not, recalling that Descartes analytic geometry had not yet been developed. But it is probable that this was implicit in Apollonius and that Galileo may have studied him.

Galileo is also known as an astronomer and also here his accomplishments were fundamental although intellectually not as impressive as his contributions to physics. No science is more intimately connected to an instrument than astronomy is to the telescope. The telescope was invented by Hans Lippershey in 1608[4] and was as such rather poor in performance allowing only a magnification of three times. Galileo heard about the principle behind the invention and was able to reinvent it himself and vastly improved allowing a magnification of 33 times. This points to another skill of Galileo, namely that of being technically adept and practical. Theoretical and practical talents are often said to be complementary to not say exclusive of each other, nothing can be more false, usually they reinforce each other. Being practical is not the same as being dexterous with your hands, but requires a solid theoretical understanding, lack of dexterity can easily be compensated for, theoretical understanding cannot. Galileo actually made a long line of inventions proving his worth as an engineer, some of them brought him money, of which he was often in short supply[5]. The same practical skill could also be seen in Newton as we will see. Now in early 1610 having reinvented the telescope he trained it on the sky, not on terrestrial objects, and made some startling discoveries. One was the Moons of Jupiter, four faint stars in the vicinity of the planets, but whose movements clearly showed that they were orbiting the planet, which also for the first time was seen as a disc. Galileo realized that they could be used as a clock showing the same time all over the Earth and hence be used for determining longitude, especially as the clock was very regular[6]. This worked out for stationary objects such as cities and other geographical markers, and hence aided accurate map making, but less so for navigation involving stormy seas, for which celestial observations with telescopes were not feasible. The final solution involving an accurate clock also under adverse seafaring conditions was not presented until the 18th century by the British clock-maker John Harrison[7]. But the discovery of the moons of Jupiter had also another deeper significance, as it actually showed that other bodies than the Earth had satellites, making the heliocentric assumption even more likely by removing some of the psychological inhibitions. An even stronger effect was to be seen in the discovery of the imperfections of the Moon[8]. Closer scrutiny revealed a surface highly irregular and marred by mountain chains and craters. If anything it showed the Moon to be a most prosaic stone in the sky. A stronger case against the classical geo-centric picture was the discovery of the phases of Venus, in fact it falsified it. The compromise suggested by Tycho of a sun orbited by planets but orbiting the Earth was for all intents and purposes indistinguishable from the heliocentric view, except psychological, and ever since the time of Copernicus the Church, meaning its leading men, had no problem with the heliocentric hypothesis as a convenient 'change of co-ordinates', to use an anachronistic but apt expression, for computational purposes. Galileo did publish his results and met with understanding among many of the cardinals, especially Barberini who turned out to be a friend and supporter. In Italy at the time the top echelon of the Catholic church, with the Pope, had a lot of power and hence prestige and authority. The fact that a lowly academic should move in such circles says something about the prestige and concomitant attention that came his way. Few academics of today enjoy such social recognition. This is something to keep in mind when reading about an ignored Galileo struggling to get attention for his ideas. Galileo was in addition to being an outstanding theoretician and a most accomplished practical man also a very skilled writer. He wrote in the vernacular and as a consequence his pamphlets and dialogues reached a relatively large audience. But being Galileo and a formidable polemicist in addition, does not always make you right. Galileo had mistaken ideas of the origin of the tides and sought to show them as consequences of the rotation of the Earth (which is just an incidental part of the story) and rejected the influence of the Moon as mere superstition and smacking too much of astrology[9]. Furthermore he saw meteorites and comets as atmospherical phenomena and vehemently attacked a man of the church who claimed, on rational grounds and empirical observations, that a comet moved beyond the Moon. The Catholic clergy was not that collection of bigots as often is presented in so called Whig accounts, instead many were highly educated and also genuinely interested in science. The strongest scientific case against helio-centrism was, as have been noted, the lack of observed parallax, which would not only imply huge distances, which caused no problems for Copernicus and his supporters, but also that the stars would be impossibly big. This was due to that they were thought of having extension in the visual field, up to 5 seconds of arcs, as measured by distances to strings of known thickness just about blocking out their light[10]. The fact that this extension was spurious and due to the optical disturbances of the atmosphere was not appreciated at the time.

Something happened. When Galileo published his dialogue in 1634 championing the Copernican system, he incurred the disapproval of the Church in spite of his friend Barberini had in the interim been chosen as the new Pope Urban VIII. The reasons for this change of heart, or rather radicalization of the position of the church have never been fully explained. Obviously it was political (which can mean many things). It has been noticed that the Pope felt humiliated by having his arguments put in the mouth of Simplico (an Italian word giving associations to 'simpleton' and in fact made by Galileo to appear as a fool). Thus the polemical spirit of Galileo getting the better of him.

The rest is history, Galileo was forced to retract, yet according to legend muttering under his breath *E pur si muove!* (still he moves), condemned and his sentenced commuted to house arrest. He lived on getting blind and under the care of his surviving daughter for another eight years. His works were not taken off the Catholic Index until fairly recently. As a consequence the Church painted itself into a corner and would then marginalize itself from the scientific revolution and the subsequent enlightenment being the target of Voltaire's curse *ècrasez l'infâme* . Why the Church decided to pursue this option lies beyond the scope of this book.

Kepler

Johannes Kepler was born on December 27 1571 in the town of Weil der Stadt, now part of the Stuttgart region. He is not known primarily as a mathematician, although he was a remarkably accomplished one, but as an astronomer providing the link between Copernicus and Newton, and as such supposedly one of the giants on whose shoulders Newton claimed that he had stood. The beginnings of his life were not auspicious. His father was a mercenary and a drunkard and abandoned his family when Kepler was five and was never heard about afterwards and presumed fallen in service in the Netherlands. But his grandfather Sebald Kepler had been a mayor in the town and no doubt saw to it that the gifted, but frail boy, got an education, after all old Kepler was of nobility and his grandson had inherited the distinction along with the blood. His mother was a daughter of an inn-keeper, and her son grew up in the inn, and used to impress travelers with his mathematical prowess as a child. From an early age he got fascinated by astronomy. Not yet six he observed the great Comet of 1577[11] with his mother, and a few years later he witnessed a lunar eclipse, which likewise made a deep impression on him. He was seen through a Grammar school, a Latin School and also a Protestant seminary at Maulbronn being for some time intended for the clergy[12], before he concluded his studies at the university of Tübingen where he was introduced both to the Ptolemaic system and the Copernican. Kepler was instinctively drawn to the latter for his elegance, although at the time the Copernican system was not very elegant. It was still in the Ptolemaic spirit using circles and epi-circles, and having reduced the number from around 70 to about 30 may possibly have been seen as an improvement. More seriously though it was not very accurate, and Copernicus, unlike Tycho Brahe, was no skilled observer and thus he could not contribute with new and accurate data. His clerical ambitions were discouraged and instead he was appointed as a teacher of mathematics and astronomy at a Protestant school at Graz at the age of 23[13]. At Graz he wrote a defense of the Copernican system as well as conceiving of a mathematical relationship between the planetary spheres based on inscribing and circumscribing Platonic Solids, to be published under the title of *Mysterium Cosmographicum* late in 1596, and sent to various potential patrons. Predictably it was not widely read but did establish his reputation as an accomplished astronomer. It would provide a life-long theme and later modified editions would appear quarter of a century later. To a modern mind the mixture between the mystical and the religious and the sober rational reasoning with a meticulous attention to and respect for facts may seem strange, not to say inexplicable. But it was not confined to Kepler, we may also observe it in Pascal and even Newton as was later revealed. In Descartes it is less pronounced but he remained a devout Catholic throughout his life[14]. For one thing there is no real contradiction between deep religiosity and a level-headed scientific approach as long as you keep their methods apart. Ultimately faith in scientific reasoning cannot be based on it, and intuition, divinely inspired or not, seems to transcend reason not only supporting it. The scientific revolution during the 17th century took place during a time of great religious upheaval, much more so than during the preceding. As far as politicians were concerned, the great religious wars may have ultimately been nothing but a matter of power, but for intellectuals, taking religion seriously, it was a time of doubt and trauma and soul searching. Scientific study is propelled by curiosity, which is a strong emotion, as strong as they get and hence very much one of passion. It is in this light one should not be surprised by Kepler's obsession with the Platonic solids and their possible implications to the nature of the world, they were all of one piece. It is not hard to detect similar tendencies in distinguished scientists of the Modern World as well.

In Graz Kepler was also introduced to a young widow by name of Barbara Müller, in fact widow twice over in spite of her rather young age of twenty-three. She had a young daughter but was also well-off due to inheritance and her own family and Kepler started to court her, presumably out of romantic desire not cold calculation. Her family was against the match due to Kepler's poverty but would eventually be forced to relent. The union produced additional children, most of whom succumbed in infancy, as has been common throughout human history. While still in Graz and the early years of matrimony Kepler came into contact with two rival astronomers Reimarus Ursus[15] and Tycho Brahe and a correspondence ensued, primarily with the latter. Soon thereafter there came to a meeting in Prague, to which Brahe had moved in 1597 leaving Denmark as a result of a disagreement with the new king Christian IV[16] and getting an invitation from the King of Bohemia and at the time also Emperor of the Holy Roman Empire - Rudolf II, to become an imperial astronomer. Brahe, whose empirical data from observations surpassed anything that had achieved by the time guarded his work carefully only allowing Kepler limited access to test his new ideas. Later after some negotiations there came to an understanding and Kepler was hired as an assistant with a regular salary. The association with Brahe did not last long due to the latter's sudden and unexpected death in October 1601[17] but as a consequence Kepler was appointed Brahe's successor now with full access to all his data. The following eleven years as the imperial astronomer would be the most productive of his life, in spite of financial troubles[18] and the concomitant domestic strain.

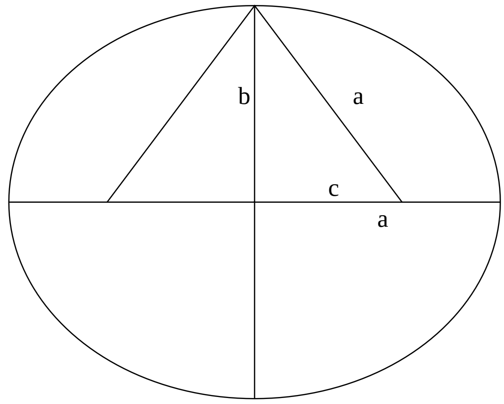
Kepler was a pioneer of modern optics publishing his *Astronomiae Pars Optica* in 1604. He worked out reflections in curved surfaces and proposed the inverse square of light intensity[19], explained the principles of a pin-hole camera, worked out the basics of parallax and apparent sizes, which were of course well-known but not necessarily documented. Interestingly, but typical of the times, he also branched out into biology and investigated the optics of the eye, being the first to realize that the image is projected upside down on the retina[20]. Optics with their light rays is intimately related to projections, hence perspectives and projective geometry. Apollonius had became known in the 16th century[21]. Dürer was under the impression that an ellipse, being the section of a cone, should be shaped like an egg[22], Kepler had a much firmer command. He thought of the parabola as the limiting case of ellipses with one of the two foci escaping to infinity, and when a line is extended indefinitely, it closes upon itself as a circle[23]. From this we conclude that Kepler had a very intimate knowledge of the conic sections. Yet, the idea of fitting orbital data with ellipses, did not come to him straight away, he initially thought that this would be too simple, only after some forty failed attempts using various egg-shaped ovoids did he explore it seriously with great success in 1605. Having determined an elliptical orbit for Mars, he boldly claimed that this was true for all planets. This is known as Kepler's First law, the second that the radius vector given by the Sun and the planet sweeps out equal areas in the same time, was actually discovered first. The third law that compares different planets came much later. The first two laws, of which the second was considered the most striking, was published in his *Astronomia Nova* in 1609[24] in which he makes a strong case for the Copernican system, arguing with commendable honesty that from observations alone you cannot distinguish them, each may be made to fit the data and give accurate predictions. The third law in his *Harmonices Mundi* 1619. Now the three laws of Kepler are fundamental enough to be stated in their entirety.

I. *A planet moves in an ellipse with the Sun in of its two foci.*

II. *The radius vector formed by the Sun and the planet sweeps out equal areas during equal times.*

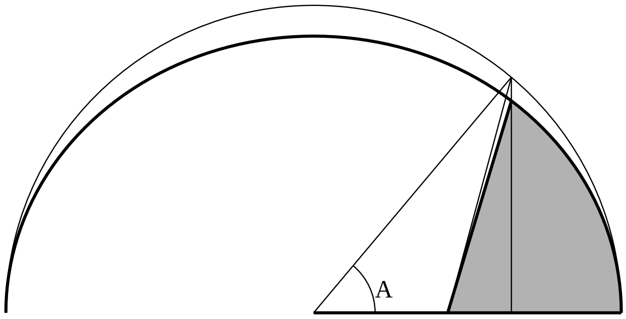
III. *The cubes of the major axi are proportional to the square of the orbital periods.*

There are many comments which may be made, the very first is that this was by far mathematically the most sophisticated model that had been presented so far. Thus the mathematical content alone, regardless of its cosmological applications, are worthy of extended inquiry. First let us dwell on the elementary properties of an ellipse (see figure below).



There are two axi, one major - the horizontal - of length $2a$, and one minor -the vertical - of length $2b$. There are two foci lying on the major axis at a distance c from the center. The basic property of an ellipse is that it is the locus of the points whose sum of the distances to two points is a constant (necessarily the length of the major axis). The two points will turn out to be the two foci. Furthermore, of minor interest in astronomy, is that light rays from one focus will be reflected to the other. Those two basic properties were known to Apollonius, and hence also to Kepler. The equation of an ellipse, using the major and minor axi as x and y respectively will be $\frac{x^2}{a^2} + \frac{x^2}{b^2} = 1$. This is of course implicit in Apollonius, but would of course not play any role in the thinking of Kepler which would take place before the development of analytic geometry. From this we see that the ellipse is formed from the circle by the vertical scaling of $y \mapsto \frac{b}{a}y$. This is of course nothing but a restatement that we can get the ellipse by intersecting a cylinder with a slanted plane. It is to be noted that the transformation behaves well as to areas, thus we can write down the area of an ellipse immediately as πab. To work out the arc length of an ellipse is something much different, and that would provide a theme in mathematics for the centuries to come with unexpected ramifications, suffices it would provide difficulties in the application of the second law. The quantity c gives a measure of the oblateness of the ellipse. To make it scale invariant we need to normalize it, conveniently done by dividing by a and traditionally $e = \frac{c}{a}$ is referred to as the eccentricity of the ellipse. It satisfies $0 \leq e < 1$. In the case of the circle we have $e = 0$ and as $e \to 1$ the ellipse approaches an hyperbola. We can also define a, b, c and hence e for hyperbolas, and then we would have $e > 1$. We can of course compute e from aa, b by noting in the figure above that $c^2 + b^2 = a^2$ by Pythagoras, and hence that $e = \sqrt{1 - \frac{b^2}{a^2}}$. For the planets e tends to be rather small, for Venus the orbit is almost circular[25] Mars and Mercury having the largest, since the dethronement of Pluto[26]. If the Sun is placed at the right focus, the right end point of the major axis will also be the point closest to the Sun and called perihelion, while the left end point will be furthest from the Sun and called aphelion.

As to the second law we refer to the figure below



If we normalize $a = 1$ and give the angle A we can easily compute the area of the shaded region to be $\frac{b}{2}(A - c\sin A)$ where A is measured in radians. If the movement is according to the second law we get the equation

$$(A - c\sin A) = kt$$

where k is determined by the orbital period T via

$$(\pi - c\sin\pi) = \pi = \frac{kT}{2}$$

To deal with that equation would at the time have been quite difficult, and computationally rather formidable, would you like good accuracy[27]

For circular orbits, Kepler's first law is immediately, the second follows from a constant central force, and for the third it is easily seen to be equivalent to the force decreasing as inverse square, which would be the starting point for Newton. Kepler did play with the idea of gravitational attraction being proportional to the bodies, but he had no notion of it as being dependent on distance.

That Kepler proposed elliptical orbits are often presented as if it would be just a matter of curve fitting. The formidable technical difficulties which he encountered are seldom explained. One should keep in mind that the data of planetary positions that are observed are given as points on the celestial sphere, the actual distances cannot be directly gauged[28]. They cannot be determined by parallax on the Earth but have to inferred geometrically and making some initial assumptions and will all be in reference to the orbit of the Earth, i.e. in terms of astronomical units, whose comparisons with terrestrial measures is another matter altogether[29]. Those are very hard technical problems, and were even more so in the time of Kepler, who had to transcend the primitive general methods available at the time. It makes his achievement so much more impressive.

In connection with Galileo's observation with the telescope in 1610 there was a correspondence between them initiated by Galileo to enlist support. Kepler was very excited about it and had a variety of suggestions and actually came up with an improved principle of a telescope using two convex lenses, when Galileo used a convex and a concave. But Galileo took no future notice of him, in particular he never commented on Kepler's law. Neither would Descartes.

There are also a host of other mathematical achievements and conjectures of Kepler which have no bearing on his astronomy. The most well-known is the packing problem, how to pack spheres in the most economic way, the solution proposed by Kepler was only recently verified by a long computer aided proof[30].

The last two decades of Kepler's life were riddled with misfortunes involving deaths of his children. In 1611 his wife contracted Hungarian spotted fever and died. He moved to Linz and two years later he married a younger woman the 24 year old Susanna Reuttinger which bore him a couple of children, some of which reached adulthood and provided a happier union than the first[31]. He suffered financial difficulties, and his mother was accused of witchcraft but Kepler managed to come up with an effective defense and have her acquitted. The onset of the Thirty Year War did not make life any easier for the Lutheran Kepler in a predominantly Catholic setting. He served for a time as the astrologer of Wallenstein[32]. He spent an ambulatory existence for many years, finding temporary sanctuary at Ulm and Regensburg where he died in November 1630. Swedish troops destroyed the churchyard in which he was buried. On his grave was the epitaph (which survived through other channels)

Mensus eram coelos, nunc terrae metior umbras

Mens coelestias erat, corporis umbra iacet[33]

Descartes

Descartes was born on March 31 1596 in La Haye a small town in the vicinity of Tours. He stemmed from old nobility, but his father was by no means wealthy although in comfortable economic circumstances[34]. His mother died a few days after his birth[35]. He was a delicate and inquisitive child pandered to by his father and his nurse and sent at the age of eight to a Jesuit school, as befitted a boy destined to become a gentleman. The principal was very understanding and encouraged the frail boy to lie late in bed in the morning and not get up until he felt ready to join his classmates at the lessons. A very liberal and flexible attitude to education seldom to be had in the supposedly more enlightened age of today. Throughout his life Descartes would be a late riser and claim that the core of his work was done in bed thinking. Hardly surprising he excelled at school[36], the work of which was focused on classical languages as well as raising him as a good Catholic[37], something Descartes would remain throughout his life despite accusations of atheism which would regularly come his way. The little mathematics he encountered excited him taking to logical reasoning as the fish to water in the words of E.T.Bell in his thumbnail sketch[38]. All in all it was a happy and relatively fruitful time and he made life-long friends, the most relevant being Mersenne, somewhat older than Descartes, and who chose a theological career and would act as a supporter and intermediary and a regular correspondent. Yet he was profoundly disenchanted by the education he had received, classical studies and the subtle points of grammar and scholastic logic he found rather sterile and not the kind of knowledge that equips you for the world around you. And as to knowledge, how much of that was actually secure? And so started in earnest the guiding principle of skepticism which would pervade his future thought.

Out of school and thrown into the real world outside (the ultimate dream of school boys) he lived for a few years the life of a dandy in Paris, the kind of life befitting his social position, but that did not prevent him from studying at the university of Poitiers, earning, according to the wishes of his father, a degree in law at the age of twenty. It was (supposedly) a life of gambling and drinking and no doubt women but after a few years he would get tired of his boisterous companions and instead seek peace and quiet for philosophical meditation[39]. But throughout his life he would be well-dressed with a rapier dangling from his hips and a magnificent ostrich plume attached to his hat. Although seen as quaint and starling to our eyes, it is in fact nothing more remarkable that at the time clothes, much more than now, were markers for social standing of immediate recognition, and as a wise man he adhered to social conventions, which tend to simplify life and free it for concerns that really matter. The times were different then also as to less trivial aspects. The career that Descartes decided to pursue, for all intents and purposes that of a mercenary, would not be a choice for an intellectual of today. Paradoxically the reason for that choice was not adventurism per se, although that certainly must have played some role, but the opportunities it provided for peace and quiet. Armies at that time did not see much action, a battle was an exception, be it a catastrophic such, most of the time was spent on the march as a swarm of locusts looking to be fed, the chores of which Descartes as a gentleman and officer, could be spared. He would all his life surround himself with body-guards when necessary and servants thus living a life of relative comfort if never one of opulence. To prepare himself for his career he went to Holland, Breda more precisely, to learn the craft under Prince Maurice of Orange, but was disappointed to see no real action (so there was a fair amount of adventurism in this ostensible frail young man after all) and actually finding indolent camp life distracting. But the most significant effect was to be introduced to military engineering and thus to be exposed to more advanced mathematics[40]. The experiments of Galileo on free fall and his exploration of the heavens with a telescope were known already.

At this time there were plenty of opportunities for mercenaries as what later would be referred to as the Thirty Years War started in Bohemia in 1618. Descartes took part in the siege of Prague which would eventually be taken deposing of the Winter King Fredrick son-in-law of the English King Charles I and whose daughter Elizabeth would play a role in his future biography. In biographies of him there are references to visions he had shortly after entering the fray along the Donau, and which he attributed to divine intervention. The upshot was that he should devote himself to science, and that he has formulated the principles of his co-ordinate geometry, realizing that those had much wider applications than just to science. Anyway the 20's were to a significant part professionally devoted to soldiering, joining the Imperial army to Transylvania, but later on returning to France and its own much more manageable civil war against the Huguenots. One may think that such a life was dangerous, it certainly was, but so was ordinary life as well. Only a minority of soldiers died in action, most from diseases acquired during camps, their insanitary conditions making them ripe for all kinds of epidemics. One surmises though that Descartes due to his rank were spared much of the vicissitudes that were the lot of the rank and file. Once failing to get a commission to the army he set out as a tourist to Rome but failed to meet Galileo, which is hardly surprising, who was Descartes at this still rather modest age? But that would change of course. During these years when he had started to develop his general methods, and in particular what would be known as Cartesian co-ordinate geometry (or analytic geometry as it was commonly being called[41]), he eventually obtained quite a reputation through his voluminous correspondence touching on his work. It was the Catholic Clergy who looked out for him and encouraged, nay pressed him, to publish. The roles of science and the church are complicated ones, which we have had plenty of opportunities to ponder, and not at all neatly reducible to an opposition between a conservative clergy ossified by adherence to an obsolete authority and a forward looking free-thinking community of inquiring minds leading us to a Brave New World. Institutions consist of individuals after all, and most intellectuals were in fact part of the church, so it was there the new ideas had to be received and nourished, hardly among soldiers and merchants. It is often assumed that the spirit of rebellion that marked early Protestantism would be more conducive to science, but the Protestant churches often turned out to be even more blatant and bigoted in their opposition to science than the Catholic. After all passion and purity of purpose are less amenable to tolerance than the more indolent and corrupted attitude that we are told characterized the Catholic Church[42]. The conflict between church and science was predominantly a political one not an intellectual, just as the religious conflicts that ravaged the 17th century were ultimately about power.

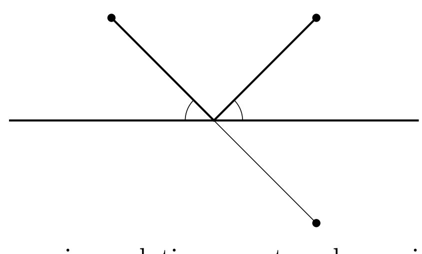
During the 30's and 40's Descartes spent an ambulatory existence in Holland, sometimes on his own, sometimes attached to a university. Holland at the time has a reputation for being a sanctuary from persecution, meaning religious one, where there resided an atmosphere of tolerance and freedom of expression. Spinoza for one found here a place to work and live. As most idealizations they do not bear up to closer scrutiny, but nevertheless they often contain an incontestable kernel of truth. There was plenty of religious bigotry in the predominantly Protestant Holland, but in any civilized society the prejudices of one group are not allowed to become uncontested. Descartes enjoyed the protection of the Orange House just as he had also acquired the benevolent blessings of Richelieu. Still his addresses in Holland were mostly secret just as they were constantly changing, and his correspondence was managed and channeled by his old schoolboy-chum Mersenne.

The intellectual tastes of Descartes were omnivorous, not only mathematics, physics and astronomy but also anatomy, medicine and biology in general. Anything was grist to his mill, once again quoting Bell. From an intellectual point of view it was an exciting time to live, and an inquiring mind such that of Descartes, was bound to discover something new in any endeavor he chose to pursue, especially if, as in the case of Descartes, his general inquisitiveness was guided by an overreaching philosophy aimed at getting the large view. So finally he was to produce his *Magnus Opus* titled *Le Monde* as a testimony to the grandiosity of his ambitions. Descartes was the quintessential rationalist setting out to prove that of all potential worlds the actual one was by necessity the only possible one. This does of course make you think of Leibniz and his doctrine of this being the best of all worlds, savagely satirized by Voltaire[43]. But at the same time Galileo was seized by the inquisition under the connivance of the Pope, a former friend and protector, and made to recant[44]. If Galileo ran into trouble for his rather mild transgressions, what about his case setting himself up as presenting what amounted to a rival to Genesis? Descartes held back. Now the logic of Descartes reasoning is not entirely relevant, when it comes to underlying politics the ostensible manifestations have little to do with it. It is quite likely, considering the different political situation in France and Holland that Descartes would have gotten away with it. Eventually he would publish another work under a lengthy title, which can be conveniently shortened to the *Method*. This is the work for which he is justly famous, and really consists of three books, the last one on geometry, with a prefatory introduction to provide their common theme, as that of being applications of a general method. His contributions to medicine, although far from non-trivial (we have already mentioned his study of the circulatory system), were later surpassed. They deserve mention, however, because they show that he was not just a pure rationalist, but was quite aware of the importance of empirical study, and more to the point, did undertake it himself. His contributions to physics, which were mostly speculative and not empirical, were overtaken by Newton who showed that his theory of cortices to account for the movement of the planets was plain wrong. As to his philosophy, which mainly boils down to Cartesian Dualism, is now, as already noted, a source of embarrassment to modern philosophers of consciousness, although it has great psychological realism. But, as cannot be repeated too often, his contributions to mathematics are the only ones of lasting values, and to those we will return in the section on geometries.

As a thinker, i.e. as a philosopher and scientist, he did not have the detachment one ideally attribute to those. He was of a prickly temperament and liable to be vain and jealous and contemptuous of rivals. In that he does not differ markedly from Newton, and many others, great and small. In the case of him and Newton a large part of the irritation they felt can be explained by being in the right and the concomitant frustration of this not being immediately acknowledged. Intellectually they tended to be head and shoulders over their contemporaries. With loyal friends such as Mersenne, Descartes had no problems, he was not a rival for one thing, it was different with Fermat, to whom we will return, and with which he had some controversies. If people presented no threat and entertained no conceit in their dealings with him, he could of course be very charming that is only human. One case in point was the princess Elizabeth, the granddaughter of Charles I, who lived in exile in the Hague, after having turned a refugee after the surrender of Prague after the siege Descartes had taken part in during his youth. A clever girl vexed by the idleness of her position and alerted to his fame she contacted him as to become her instructor. Maybe because of social vanity, he had in his youth been somewhat of a dandy, and the temperament may not have worn off completely, he assented and a correspondence ensued not without interest to posterity. More momentous though was the call of real royalty - the Queen Christina - daughter of the legendary Gustavus Adolphus and head of the victorious state of Sweden just after the Treaty of Westphalia, when that country was at the zenith of its prestige. He received a call to Stockholm to become the personal philosopher of the young Queen[45] who had already started collecting famous scientists to her court. He was taken to Sweden in the fall of 1649 but expired a few months later from pneumonia. This is traditionally taken as Sweden's only interference with philosophy, killing off one of its star representatives. The cause of his death is commonly attributed to the harsh and unfeeling treatment given to him by the young insensitive Queen of an iron constitution, involving him to be at her call at an ungodly hour in the dark and cold northern winter, he of such a delicate constitution and accustomed all his life to be a late and indulgent riser. The real story is quite likely to be more prosaic. Life was hard irrespective of latitude and the philosopher could as well have contracted pneumonia back in Holland or France. Anyway Death is an awful thing and pneumonia may be one of the more merciful ways to go[46].

Fermat

Fermat was born in August 1601 and spent most of his life as a judge in Toulouse. Mathematics was done on the side, or perhaps his professional life was done on the side as to support him, reserving his passion for what he cared most about. Anyway his work as a judge cannot have been too irksome and time consuming, provided of course if you knew how to deal with it. As Bell notes in his sketch, social interaction as a judge was strongly discouraged lest the judge would lose his detachment. He did work in optics, known especially for the principle of least time which 'explained' the laws of reflection and refraction, and which later would serve as an inspiration for the principle of so called least action (or more accurately, if somewhat pedantically, stationary action) championed by Maupertuis and playing a crucial role in the modern reshaping of classical mechanics by Lagrange and Hamilton.



He was also a pioneer in analytic geometry along with Fermat and anticipated the calculus which later would be developed by Newton and Leibniz, in particular his systematic way of dealing with tangents. With Descartes he corresponded and knew of course Mersenne who played a pivotal role in the epistolary networks which developed and also ran a salon which would serve as an inspiration for scientific socities which would arise later[47]. Relations between Descartes and Fermat was hardly surprising rather tense, Descartes seeing him as a rival, and thus ready to accuse him of plagiarism. More fruitful was his correspondence with Pascal which lay the foundations for modern probability. But it is as a pure mathematician he is mostly remembered, and as such he may be considered the foremost of the 17th century. Fermat was interested in Diophantine equations and studied an old Latin translation of the works of Diophantus an interest which must have been rather unique at the time[48]. Fermat did not prove everything he claimed, but it is of course hard to know exactly what he proved, much of his *Nachlass* may have been lost even before his death, but they served as a legacy and challenge to the mathematicians of the 18th century. The most fundamental is what is known as 'Fermat's little theorem' to the effect that $n^p \equiv n(p)$ for any n and prime p. There are very easy proofs of it, but that does not mean that it is easy to prove[49]. Another version of it is that if $p \not| n$ then $n^{p-1} \equiv 1(p)$. The crucial insight is that in this case the map $x \mapsto nx$ is an isomorphism among the non-zero residues of p, or in a modern conception, the non-zero residues modulo a prime form a group. Once this is understood, the sequences $1, 2, 3 \ldots p-1$ and $n, 2n, 3n, \ldots n(p-1)$ form the same residues and hence their products are the same, and as the second is n^{p-1} and the products are non-zero we can draw the desired conclusion. An alternative proof in the group theoretic vein is to invoke Lagrange theorem to the effect that the order of a subgroup divides the order of the group, shown via reasoning by cosets and proved implicitly by Lagrange before the notion of a group had even been made explicit. This illustrates the fact that in a proof the idea is paramount not the formulation of the claim of which the proof purportedly gives a justification. The consequences of an idea are manifold and can thus be formulated in many different ways. If you want you can see this as an illustration of the supremacy of Plato's forms over their sensuous manifestations. Fermat's little theorem may be simple but it is more fundamental than anything else he did[50]. Fermat's study of Diophantus yielded some observations that the diophantine equation $x^3 = y^2 + 2$ only has the more or less immediate solution $x = 3, y = 5$. Such equations would later be part of the arithmetical study of elliptic curves[51]. His claim that primes of the type $4n + 1$ can be written as the sum of two squares[52] (in a unique way up to trivial variations) constituted an entirely new way of looking at numbers and relations, and the success of it is due to the fact that we have here the beginnings of the algebraic study of quadratic forms, which you may think of originating geometrically through Pythagoras theorem. That the product of two sum of squares is another sum of square had been known for some time, also in non-Western traditions through the algebraic identity[53].

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$$

What does it mean? By hindsight we see that this is a consequence of multiplication of complex numbers, in fact using the multiplicativity of the norm $a^2 + b^2$ of a complex number $a + ib$. This we can see as an explanation, but to verify the identity as such, there is no need of complex numbers, but then the activity is one of blindness. In fact a modern proof of Fermat's theorem about sum of squares use complex numbers, more specifically the unique factorization of Gaussian integers, but this is not the way Fermat convinced himself of it but through what he called the principle of infinite descent[54]. Would it not be true, he would be able to find a smaller counter-example and this cannot be done indefinitely no matter how large a number you start from[55].

However what Fermat is most known for is that the diophantine equation $x^n + y^n = z^n$ cannot have any solutions for $n > 2$[56]. Fermat wrote famously in the margins of his copy that he had a wonderful proof of the fact but that the margin was too narrow to allow it to be written down. This, maybe the most famous marginal note (about a missing marginal note) in history, was discovered by his son after his death[57]. The consensus is that Fermat simply could not have had a proof because the final proof of Fermat's grand theorem evaded generations of mathematicians for over three hundred years, but this is not a proof. However, there is a a surviving proof by Fermat that the equation $x^4+y^4 = z^2$ has no solutions, which clearly implies the non-existence of solutions to $x^4+y^4 = z^4$. This proof can hardly have been written down after the margin note, unless he had discovered a flaw. But had it been written down before the note, the spontaneity of that note would be hard to explain, it is as if he for the first time pondered the question and had a brilliant flash of inspiration.

Fermat's theorem would play a central role in mathematics ever since, and one may argue why it did? Mathematically it presented a clear challenge, because there were no obvious way to attack it. It provided the same motivation as Mt.Everest provided for rock climbers. It was there! Nothing really hinges on the theorem, it does not have any applications, not even mathematical ones. Many mathematicians were disdainful of it, in fact Gauss. Riemann, Cauchy, Weierstrass, or any number of distinguished mathematicians ignored it. Gauss actually disparaged it, one could easily write down any number of equally intractable equations. But in algebra it actually provided inspiration and many features of algebra was developed because of it. Using Gaussian integers and Eisenstein numbers (generated by a primitive cube root of unity) one can actually prove it fairly elementary for $n = 4, 3$ respectively. Kummer thought he could generalize this approach but the proof was flawed rings assumed to be unique factorization domains were not, but as with many failures, they are more interesting and fruitful than successes. Fermat's theorem, because of the frustrated efforts to prove it acquired a nimbus. That nimbus is to be thought of as cultural not mathematical. Norbert Wiener claimed that any mathematician worth his salt should try at least one of the outstanding conjectures, the four-color problem, Fermat's theorem, or the Riemann hypothesis. He himself had tried all three. Would you prove any of them you would acquire recognition and fame. The first two are easy to understand for people with no mathematical education, the Riemann hypothesis, easily the most serious and mathematically meaningful challenge remains very much a challenge, but not one to be appreciated by the public.

The eventual resolution of Fermat makes up a fairy tale of sorts. In 1964 (about the same time I encountered Bell and mathematics beyond that of the school) a ten year old boy checked out a book out of a local library. He got fascinated by the problem, one he could easily understand. He decided to solve it (and become famous). This is normal for young boys. He pursued mathematics and became very successful as a number theorist studying elliptic curves. In the interim more and more cases had been settled by various tricks. This showed that one should not think of it as a series of ordinary Diophantine equations, one for each n but as a more complicated one, in which n too was one of the variables. It seemed likely that for each specific n one may with increasing labor solve it. From a logical point of view it could be that it was undecidable, that there was no uniform argument to cover all the exponents n[58]. In other words a not well-formed problem leading into an infinite labyrinths of *cul de sacs*. But then in the early eighties a German mathematician Frey noticed that if it had a solution one could use it to construct a certain peculiar elliptic curve which would not be what the experts called Modular in spite of conjectures to that effect. This statement was put on firm grounds by K. Ribet in 1986 and did generate some excitement in the mathematical community, maybe it would yield to modern sophisticated technology after all. The young boy who by this time had become a young man in his early 30's took it *ad notam* and closed himself off in his chamber for seven years attacking it. Strangely enough during this time of living the life of an eremite he also found the time to marry and sire a family, In 1993 he gave a lecture at the Newton Institute at his Native Cambridge. At the end of a series of lectures he noted: by the way, this proves the Fermat conjecture. His name was Andrew Wiles, people brought out cameras and soon enough he was known to the wider world, probably the most commonly known mathematician. There was a hitch, there was a mistake, he had not really proved the full classical conjecture on modularity, he had to go back and fix things up, now in the glare of general attention. With the help of a student (Taylor) of his he managed to prove a weaker version sufficient for his goal, a year or so later. The rest is history as they say. Working it all out had nothing to do with the equation *per se* once the connection had been made. But thanks to it all, Fermat has also become, if not a household name, something people feel they should know about. And the story also shows that still individual effort counts, in fact it is what it takes, and working in isolation, once you have your basic mathematical education in place, can work, maybe even be necessary.

Pascal

Pascal was born in Clermont, Auvergne June 19, 1623. His father was a highly placed judge, in fact chief of the courts in Clermont, and of no mean cultural and intellectual distinction, whose standing with the authorities, i.e. with Richelieu waxed and waned as such things tend to do. Pascal lost his mother when he was four years old, but as a compensation he had two talented older sisters who took a strong interest in him and played important, if not always fortunate rôles in his life. Thanks to them we have a good documentary record of his childhood. The family moved to Paris when Pascal was seven as a consequence of his father's advancing career (he would end up as state councilor) and from then on he enjoyed paternal teaching. Pascal was physically frail but intellectually very precocious, so advanced in fact that his father was worried that such mental energy would be too much for his poor frame to contain. As a consequence he initially banned his son from mathematics, but when he discovered that he had nevertheless picked up some on his own, such that the angular sum of a triangle was 180^o he relented and instead rejoiced at his mathematical gifts and gave him Euclid's Elements to study, most of which, according to his sisters, he had already discovered for himself. As a teenager he went beyond Euclid and Apollonius and wrote an extensive treaty on conics discovering at sixteen his celebrated theorem about the geometrical significance of having six points on a conic, which we will return to on the section of Projective Geometry. Pascal was a great mathematical prodigy among the foremost on record, so great indeed that he would not really live up to his promise, the reason being religious diversions and digressions which gradually would swamp him altogether. As a consequence he is the only great mathematician who is not primarily known as a mathematician, his religious and philosophical writings, such as his *Pensées*, have thus attracted much more attention than his mathematical works, morbidly confused and mystical as the former may be, but accordingly are the rewards of this world distributed. Between his religious brooding eagerly abetted by his sisters and devoted to the teachings of a religious fanatic - Jansen [59], he nevertheless was able to do some work of more lasting value. As to physics his work on pressure was fundamental, and the S.I. unit for pressure is named after him. The fundamental work was done by the Italian Torricelli (1608-47)[60] who had invented the barometer and explained the nature of atmospheric pressure in raising a column of water only so high (and a column of mercury so much less so[61] furnishing not only the principle of the barometer but making it feasible). Pascal had noted that the barometric pressure should decrease with altitude corresponding to the loss of weight of the column of air above[62]. The experiment was actually performed and provided a confirmation. As a result he was embroiled in a controversy with Descartes who accused him of plagiarism. In addition to the experimental work he also constructed one of the first calculating machines he tried to foist on many institutional bodies, including that of the Swedish court, but to little avail. Anyway it was significant enough to justify putting his name on one of the extant computer languages [63]. Pascal also did some brilliant work on the cycloid, to which we will return, but more of the nature of establishing his brilliance than breaking new ground. However, the work done with Fermat, laying the foundations for modern probability theory may be considered his most solid scientific legacy.

Antoine Gombaud, is often described as an aristocratic inveterate gambler, but this is unfair. For one thing his title 'Chevalier de Méré' was an invention, and he was most known as a writer and retrospectively a predecessor of the Enlightenment, and incidentally not an unable mathematician. Nevertheless he moved in salon circles and it was inevitable that he would encounter gambling and become fascinated by the problems generated by the activity. He enlisted the help of Mersenne who kept a salon and through him catching the attention of Pascal and Fermat. The classical problem is the one of points. Supposed that two players are involved in a game consisting of a number of rounds each of which both have equal probability of winning[64]. Both players contribute equally to a pot and decide that the player that first makes a certain number of wins get the entire pot. Now assume that the game is prematurely aborted due to some unspecified external reasons, how should the pot be fairly divided among the two players?

First this is not a mathematical problem *per se*. What is meant by a 'fair' redistribution of the pot? In the game it goes wholesale to the winner, why should it not do the same in this case? And if so it seems fair that the one who has won the most rounds should get the entire prize? What about if they have won an equal number of rounds? Flipping a coin? That would be a post-game move. The idea of fairness seems to imply that the pot should be divided somehow. In a real game there is one clear winner, but in an aborted one, the future course could have taken any number of ways. In real life there is truth and falsity, but in a hypothetical one, black and white admits of many shades of intermediate gray. Inherent in the assumption of 'fairness' there is the idea that the future is not determined, it can take many different courses, some in which player A is winning, others when player B comes on top. If most possibilities point to A as winner, it is but 'fair' that he (or she?) gets a larger share of the pot. The real difficulty of the problem is to make it amenable to mathematical reasoning. Once you have set it up as a mathematical problem, its solution could be hard or obvious, but this is a matter of technology and not controversial. The controversy concerns the setting up of the mathematical model, as we would say nowadays, and this is always the source of controversy in all applied mathematics. If the results are unsatisfactory, mathematics is often blamed, but that is surely silly. It is the choice of mathematical model which is to be blamed. The example is also very interesting in breaking new grounds. Formerly applications of mathematics had been rather straightforward involving simple and elegant principles, indeed so that physics as articulated in mathematics became almost indistinguishable from mathematics. Physics such as mechanics could in principle be axiomatized as in Euclid and thus reducing mechanics to mathematics as an extension of Euclidean axiomatization of space, but now involving movement, i.e. time. This sounds simple in theory but harder in practice. When you solve problems in Euclidean geometry you often resort to a visual intuition which may be hard to formulate. When it comes to mechanics, there is also a mechanical intuition which seems more subtle. People often make egregious mistakes in mechanical reasoning they would never do in a purely spatial setting as in elementary geometry. But now mathematics is called upon to do service in the social world, and the notion of 'fair' only makes sense in a social setting. Numbers, i.e. quantification, now enters the quotidian world, and the important question is whether those numbers are 'real' numbers and not 'pseudo' numbers, as touched upon in the section on Galileo. In order to appreciate the subtlety of the question we need to look at preliminary attempts at solutions.

Already in a text book from 1494 authored by a certain Luca Pacioli (1447-17)[65] addressed the problem and suggested that the pot is divided according to the number of rounds won. This was criticized as counter-intuitive, among others by our old friend Tartaglia. Just think if only one round has been played then the winner of that takes all, while in a real game with many rounds, the winner of the first has only a marginal advantage. Thus according to Tartaglia we should also take into account how many rounds are required to win. His proposal was to divide the lead of the one ahead by the number of rounds and base the division of the pot on that, whatever that would entail[66]. But Tartaglia despaired that there would be a fair solution but that whatever stratagem which was proposed would be cause for litigation.

The new point of view of Fermat and Pascal was to focus not on the past but on the future. Because what was at stake after all was the distribution of possible scenarios, i.e. subdividing the future. Fermat's approach was to list all possible scenarios in the following way. If player A needs a wins to reach the goal and player B needs b wins, then if $r = a + b - 1$ rounds are played at the end one and only one of the players has reached the goal. This is because $(a-1)+(b-1) < a+b-1 < a+b$. Now there will be 2^r possible outcomes, some of which A wins, some of which B wins, and we just count the proportions and distribute the pot accordingly. This is a purely mathematical problem, which actually can be solved in each case by a listing of possibilities. Thus we are confronted with two essentially different problems. The first is to gauge the relevance of the model, does it conform to our notion of 'fairness'; the second is to make the actual computations involved feasible, which in many cases boils down to clever combinatorics. As to the first we note that in many cases a win of either party will have been achieved before the full number of rounds r has been played, but we will not consider such a sequence of rounds finished but consider all possible continuations, even if of no interest. Thus they should all be counted and thus giving the appropriate 'weight' to the total tally. Is this 'fair' [67]? In a sense one may make an empirical test to seewhether this is reasonable. Let two gamblers play say a million games, and for each game we make a note in passing after say a certain fixed number of rounds. In the end we look at the outcomes for each fixed distributions and see how it complies with the predictions. This would give us a coupling to the real world, but it would be connected with two weaknesses. If we want to know if a coin is fair we can toss it N times and see if head turns up $N/2$ times. It seldom will of course, but if we allow a certain tolerance we can draw a conclusion that it is reasonable to assume so. But even a fair coin can come up head ten times in a row. Such series are exceptional and only occurs once in 2^{10} cases for fair coins. If we assume that our series is not exceptional, we would conclude that the coin is not fair. But we cannot be sure of that, so make us do ten sequences of ten throws. Most of those ten throws we expect not to be exceptional. So what is happening is that we consider probabilities of second order, we are not just checking that the coin is fair, but that the throws are fair. And so on. This is the weakness of the statistical method, we cannot say that this has that and that probability, only that it has that probability with a certain probability, and this meta statement is also to be qualified by a probability and so *ad infinitum*[68]. The basic idea is that in the short run very unlikely events do not occur (while from a meta-physical perspectives in the long run every possible event will occur)[69]. The second drawback is that you cannot prove general statements, only finite subsets of statements, and the more complicated, the more tests are needed. But this is the condition of the empiricist who has to learn to live with it. Still in this simple example it is not too hard to convince yourself that the predictions made by Fermat should be accurate, meaning fair in the sense of true and false. But from this to conclude that the corresponding distribution is fair is more or less a convention involving the identification of the two notions. And in particular people who are not convinced of the first 'fairness' will never be convinced of the second, and be ever so ready, as Tartaglia warned, to bring up litigation.

But now we come to the purely mathematical part, how do we in practice compute the distribution of outcomes? If $r = 30$ to write down a billion different cases and count by hand is no longer feasible. It is here Pascal steps in. If we set $E(r, s)$ to be the fraction A is entitled to if needing to win r times against B's s times. Then if we proceed one more round the situation will be either $(r-1, s)$ or $(r, s-1)$ both with equal probability. Assume that we know the fair distribution in each of those cases, then the fair distribution $E(r, s)$ for the case (r, s) should be the average of the two, thus we get

$$E(r, s) = \frac{1}{2}(E(r-1, s) + E(r, s-1))$$

This gives a recursive way of computing which is very efficient, and it also gives a new way of looking at the distribution of possible outcomes which may be more convincing than that of Fermat[70]. The reader may be struck by the similarity of Pascal's triangle, which is of course not a coincidence. In fact Pascal exhibited his triangle in this context, but the triangle was known long before Pascal on other mathematical traditions. Giving the set-up above it is elementary to express the $E(r, s)$ in terms of binomial coefficients[71]. But the most important consequence which went well beyond the rather trivial example, was the notion of expected value, in this case exemplified by $E(r, s)$. Together with the Dutch Christian Huygens Pascal would develop this further. And it prompted the famous Pascal's wager that claimed that the rewards of faith in God are very large, so large indeed that the expectation was high, even if the probability of the existence of God was very low. One may say this is one of the few instances when Pascal mixes religion and science, and as such one may suspect it was made in jest. Normally Pascal made a clear distinction between religion and science and warned against confusion. In science one uses reason but in religion oneas to resort to authority. This may at first seem a bit strange but consider the problems of values those can never be arrived at through reasoning, because that would make them instrumental for ulterior purposes, which in their turn will depend on tacit values[72]

Geometries

Euclidean Geometry

The geometry with which we are most familiar is the Euclidean geometry axiomatically codified in Euclid. We feel that it is the geometry of real space, although the basis for this conviction is not entirely clear. Kant claimed that it was inherent, i.e. inseparable from the way we huan organize spatial data. Thus it is important to understand that Kant did not express a belief on how real space was fashioned, famously he claimed that we could never have actaul knowledge of 'das Ding an sich', only the way we comprehended the thing. The actual part of space we can fathom is very small, but nevertheless typical. In the proofs of Euclid there is the tacit understanding of rigid motions, i.e. translations and rotations, and combinations thereof. They form a group, although the concept was not recognized and formalized at the time, but the composition of operations, is something people have always done and understood. The notion of congruent triangles, central to the methodology of Euclid, is in modern parlance intimately connected with the group of rigid motions, belonging to the same orbit. But what is peculiar to Euclidean geometry is the notion of scaling and the concomitant idea of similarity. Two triangles can have the same shape, i.e. the same angles, but not being congruent. This provides an exception to the general rule that if any three of the six parameters (the three angles and the lengths of the three sides) are known, the other three are determined. Hence the notion of similar triangles, or more generally of similar figures, and the possibilty of scaling. One may form faithful models of any part of the space, in particular maps. Thus if we can imagine a small part of our space, we can equally well imagine a much larger part. Thus by extending our imagination we can in principle go beyond any limit. Size is not an intrinsic part of space. Thus there is no natural unit in measurment, any unit will in principle do, and the choice is necessarily one of convention based on convenience and having nothing intrinsically do with geometry. The 'thumb' and the 'foot' are convenient measures for men going on their business in everyday life, just as elephants, would they develop a civilization, might use their trunks. Then if you work with the solar system an astronomical unit, the distance to the sun, is a very convenient unit of length, through which much else in the system can be expressed, such as the distances between the sun and the other planets. In many ways it is of less import to know 'currency exchanges' i.e. to relate an astronomical unit to a more terrestrial one. In fact that was done fairly late in astronomy. The nautical mile is another such unit of length, in spirit the same, as will presently be explained, as the parsec used in intestellar astronomy. Angles are profoundly different from lengths, they do not change under scaling. There are natural units for angles, in the sense that such units can be expressed geometrically. We have notions such as one revolution, adding angles at a common vertix until they all fit and leave nothing left, and thus of a quarter of a revolution, the notion of a right angle, which can be defined in any number of ways, and is so in Euclid. Now what you want to call that notion numerically is a matter of convention. According to the Babylonian tradition we speak about 90^o while as mathematicians we think in terms of radians, thus giving the numerical value of $\frac{\pi}{2}$. The reason for this is due to the convention of the unit circle, which in practice means that instead of speaking about one revolution, we think of it as of length 2π. This turns out to be very convenient, as once we pick a unit of length we automatically get the unit circle, and hence a correspondence between angles and arcs, whose lengths (whatever that really is) correspond to the angles. The real advantage of this convention only occurs when getting the power series expansions of sines and cosines (which themselves are conventions). When we think of a right angle as $\frac{\pi}{2}$ the real number π is a mere symbol. We seldom think of it as 1.5708.. that would often be inconvenient and hardly illuminating. In fact typical angles would be rational multiples of π which will be the same set as rational numbers of degrees, and in fact angles whch correspond to roots of unity.

Now the relations between lengths and angles make up what we call trigonometry which is not a study with which Euclid is concerned. It is motivated by specific practical problems, and as such can be thought of as applied mathematics. Consequently spherical trigonometry developed earlier than planar trigonometry, because the applications were astronomical and later on navigational. We will have occasion to return to this in the next session.

One of the central results in Euclids geometry is the Pythagorean theorem. This one, especially through one of the proofs given, is intimately related to similiarity and forms a backbone of Euclidean geometry, although its significance is not that easy to make apparent, except by hindsight and experience. This is another topic to which we will return.

One psychological effect of Euclidean geometry (which may have delighted Kant) is our obsession of extending any ray (directed line) indefinitely. When told that the universe is finite,we cannot comprehend this in other ways than to believe that the extension is being halted by a boundary, and asking ourselves what lies beyond it. There is as we all know another solution to finite yet boundless space.

Spherical Geometry

As noted the Greeks were not ignorant of spherical geometry, it was the geometry of the heavens, but only accessible by sight. Its exploration was to a large extent limited to the demands of astronomy and thus chiefly confined to spherical trigonometry.

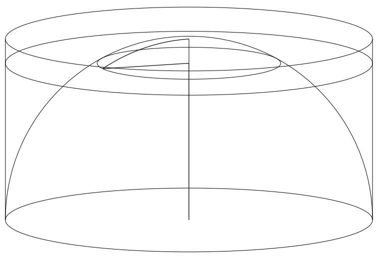
A sphere can be thought of in two ways. Either externally as a ball, a subset of Euclidean space, or internally, seen from its center. A ball is a physical object you can turn around in your hand, unless it is too big and heavy in which case you can crawl around on its surface. As a physical object it has dimensions, it is big or small. But a sphere viewed internally is no physical object, it is in fact not necessarily situated in space, but may, as we discussed earlier lying beyond space. The sphere we know most intimately by sight is the sphere of vision. Mathematically we can think of it as the space of all directions, and as such it lies beyond space. If we draw an external picture of it we have a sphere and its center. The center is supposed to be the position of the observer. If the observer moves, he or she will still be in the center, the sphere is supposed to be infinitely far away. Thus the point at the center represents all of space, and what is in between has no meaning whatsoever. The picture is just a model, a metaphor, and like all models and metaphors it does not survive being taken literally. There will inevitably be features of it which has no relevance and need to be ignored. To take a metaphor literally is to render it silly.

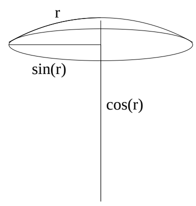
Astronomy started as a science when it was realized that the objects on it were part of our physical space. That we could start asking meaningful questions about them as to their distances to us, and hence contemplate traveling to them. Already the Greeks asked questions as to the distance to the Moon and there are a host of different methods of deciding it based on standard Euclidean space. Already Ptolemy had a good estimate. The distance to the Sun is far trickier (i.e. to make a conversion of am astronomical unit to terrestrial units)[73] and for a long time there were no indications that the stars would be part of physical space and not just dots infinitely far away, but the latter possibility seems never to have been entertained. Thus no object which we can see on the sky, not even faint distant galaxies, are expected to lie beyond space literally on the celestial vault. Is it just a mathematical abstraction, a Platonic form, with no physical existence. But is this not a tautology, having defined physical existence as residing in physical space, thus having dimensions and distances? Is physical existence the same as existence period? This mathematical abstraction serves a very important role, and a way of catching it as a physical object is to pay in particular attention to bright distant objects which can be thought of as fixed reference points on it and thus furnish us with a rather exact model of the Platonic sphere in the world of the senses.

Seeing a sphere from the inside is a very different experience than from seeing it from the outside. This presupposes that we try to shed our preconceptions derived from later [sic] experiences with physical spheres, which force us to imagine ourselves being in a center of a sphere, which we at the same time envision from the outside. To the innocent eye a great circle, view internally is a straight line. After all a great circle is the intersection of a plane through the origin. Admittedly it is hard to imagine two straight lines intersecting in two points without somehow being curved. The point is that our field of vision is local. We can only see some small part of the sphere and concentrate on it at one time. To get a wider view we must let the eye scan the sphere. Supposedly our field of vision is actually larger than a hemi-sphere, so in principle we whould be able to apprehend two anti-podal points at the same time, and hence get a real feeling for them. But our command of the margins of our vision is weak, in fact at the very margin we can only detect moving objects not stationary, and what we detect are not the moving objects per se, only their 'movements'[74]. In Euclidean geometry we solve the problem of structures too large for our field of vision to take them in, by scaling them to a manageable size. This is not an option in spherical geometry. So what we do is to externalize the whole situation by looking at a physical model, then we can see everything, but of course in so doing we miss the point. Intrinsic geometry should be enjoyed intrinsically.

> The fact that the spherical field of vision should be identified by a physical sphere is not obvious. If it could be apprehended only visually it may not be the case. In order to see the entire vault (or more precisely our field of vision) we need to crane our necks, actually to turn around, as we famously have no eyes in our necks, thus to explore it muscular movement is necessary after all. It is this effort, I believe, which makes us make the transition. An indivudal his head since infancy fastened in a vice and strapped to a chair, may not be able to make the mental transition. We see here how human cognition meets geometry.

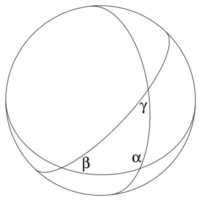
The impossibility of scaling means that we can define canonical units of lengths, meaning units we can define intrinsically geometrically. One obvious choice is the length of a great circle[75]. Thus distances in our field of vision should be measured by angles. The angles given by the two lines though the center and eminating from the points. To say that the Moon is as big as a six-pence does not make sense, but it makes sense to say that it extends half a degree[76]. It is very convenient to set that length equal to 2π. Thus we have a rather peculiar situation on a sphere. Lengths and angles are measured by the same units! Straight lines are of course given by great circles, and small circles by intersections of planes not through the center. The circumference and area of a circle with radius r will be given by $2\pi \sin r$ and $2\pi(1 - \cos r)$ respectively. When the radius is $r = \frac{\pi}{2}$ the circle will be a great circle of length 2π and area 2π. The first is obvious from the figure below, the second from the same figure and the theorem of Archimedes.





For small r we get the approximations $2\pi r$ and πr^2 with errors on the order of $-\pi r^3/3$ and $-\pi r^4/12$ respectively, which accords with the Euclidean case[77].

The Greeks were hardly aware of those discrepancies, and although the formulas are obvious enough, there would have been little incentive for them to note them down, as circumferences of circles on the celestial sphere would be of no interest to them for astronomy. The same goes for the next striking theorem which gives a relation between the angles of a triangle and its area. For the elegant formulation below it is important that we have the normalization above that gives a simple relation between angles and lengths.



Theorem *Given a spherical triangle Δ with angles $\alpha, \beta\gamma$ then its area $\mu(\Delta)$ is given by its angular excess.*

$$\mu(\Delta) = \alpha + \beta + \gamma - \pi$$

The proof of this is surprisingly simple, and would certainly have been accessible to the Greeks, but it is doubtful whether they knew about it, no trace of it has been seen. Once again, areas of spherical triangles would not have been of interest to them.

Before we present the proof how would you go about it had it appeared on a calculus exam? You know much more than the Greeks did and you know various ways of parametrizing a sphere and use those to compute the surface areas of various parts of a sphere. But would you be able to compute the area of a spherical triangle in this way. How should you choose the parametrization and how would this translate the triangle? Would you have to compute integrals that allow no simple expressions? Maybe it would be doable if the parametrization is clever enough, but how would you find such, other than by trial and error, and if so would it not take a very long time? And success would not be guaranteed. The problem of mapping a sphere onto a plane, which is part of cartography, with all its metrical distortions, is a fundamental one and provides the entry point to differential geometry, and which we will briefly consider below. Sometimes knowing too much may be a detriment, it certainly would be in this case encountering the problem on a calculus exam, where you expect to use certain powerful methods.

What about a simpler case? Consider the triangle that is formed by the equator aand two meridans ninety degrees apart. This is the standard example which is presented to convince people that the angles of a spherical triangle does not add up to 180^o, because in this case all the angles are right and hence the angular sum is 270^o or if you prefer (as you should) $\frac{3\pi}{2}$. Now in this case it is clear that we can tesselate the sphere into eight such triangles, and as the total area of the (unit) sphere is 4π, each triangle has area $\frac{\pi}{2}$ and indeed the formula above is verified in this case.

Now what would happen if the meridans are α apart, then in general you cannot tesselate the sphere with a finite number of those, but clearly you can figure it out anyway, as the area of the triangle should be proportional to α as if you subdivide *alpha* the areas add. In fact the area should be α and once again you verify the formula. Now it would be more natural to consider the areas bounded by two meridans, which will be twice that of the triangles, sso those are given by 2α. And why meridans? Any two great circles meeting at angles α and $\pi - \alpha$ divide the sphere in four regions of areas $2\alpha, 2\pi - 2\alpha$.

We are now ready to make a simple computation. A spherical triangle Δ determine three great circles (the extensions of its sides) and hence three hemi-spheres, each containing Δ (a great circle determine two hemi-spheres and we make a pick which one to choose). The union of those hemi-spheres is the entire sphere minus the antipodal image of Δ (the intersection of their complements). Now what is the area of the union of those hemi-spheres? It is clearly given by the sum of their areas, but then we count the points in the intersections of both twice, and their areas should be subtracted. But if we do that we subtract the points in their intersection once too many and we have to add that area again. In combinatorics this is known as the exclusion-inclusion principles, and we have $\mu(A \cup B \cup C) = \mu(A) + \mu(B) + \mu(C) - \mu(A \cap B) - \mu(B \cap C) - \mu(C \cap A) + \mu(A \cap B \cap C)$. In our case we know $\mu(A) = \mu(B) = \mu(C) = 2\pi$ and the area of $\mu(A \cap B)$ is given by the angle γ between the two great circles, which is the same as the angle γ that occurs in Δ. Putting everything together we get

$$4\pi - \mu(\Delta) = 3 \cdot 2\pi - 2(\alpha + \beta + \gamma) + \mu(\Delta)$$

Solving for $\mu(\Delta)$ we indeed get it to be $\alpha + \beta + \gamma - \pi$.

How come it is so simple? What could the principle be that lies behind it? Now if you have a triangle and subdivide it into other triangles it turns out that the angular excess is additive. This can easily be seen from a combinatorial argument. In fact if a triangle is subdivided into triangles and there are i interior vertices and e vertices on the edges there will be a total of $2i + e + 1$ triangles. Then it follows that the sum of the angular excesses will be given by

$$\sum_i \delta_i - N\pi = \alpha + \beta + \gamma + 2\pi i + \pi e - N\pi = \alpha + \beta + \gamma - \pi$$

where we for each interior vertex collect all the angles associated to it, which add up to 2π and for an edge vertex the corresponding sum will be π and finally for each associated with an original vertex of the triangle the corresponding angle[78]. The significance of all this will be discussed in a future lecture relating to the works of Gauss.

However, the Greeks and their Arab successors knew a thing or two about spherical trigonometry. First the spherical form of Pythagoras for a right-angled triangle with hypothenus C and sides A, B is given by

$$\cos(C) = \cos(A)\cos(B)$$

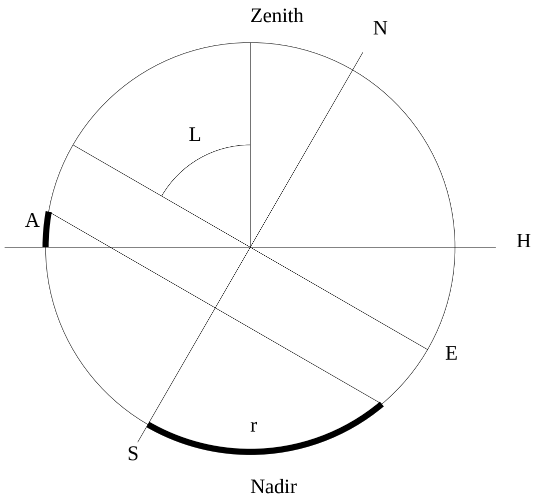
note that we are taking cosine of the lengths of the sides as those are actually angles and this, unlike the Euclidean case is very natural. Now as the sphere can be embedded in 3-dimensional Euclidean space, any statement of spherical geometry can be reduced to one of (3-dimensional) Euclidean geometry. Whether this is cheating or not is a question of taste. Clearly one should be able to set up a set of axioms for spherical geometry and prove it in the deductive way, but I doubt that the Greeks ever did that systematically, at least not in any surviving document. But they did work intrinsically on the sphere once they had enough theorems to build on. First we note that if A, B and C are small, we can replace the cosines with $1 - \frac{A^2}{2}$ etc, and then we get $C^2 = A^2 + B^2$ from the multiplicative identity, showing that for small values, we have a good approximation of the Pythagorean theorem, or equivalently small parts of the sphere are Euclidan in character. The actual errors can easily be worked out by the reader.

Now let us denote the vertices of the triangle with α, β, γ with α opposite A etc and think of them as unit vectors. With modern vector calculus we can express the statement above as $< \alpha \cdot \beta >=< \alpha \cdot \gamma >< \beta \cdot \gamma >$ under the condition of orthogonality which can be expressed as $< \alpha \times \gamma \cdot \beta \times \gamma >= 0$. Can we derive the former from the latter? This would be a very anachronistic way of doing it, the point being that anyone mathematically literate but innocent of spherical trigonometry may be startled by the assertion. Of course doing it formally by manipulation would not really contribute anything to the understanding [79]

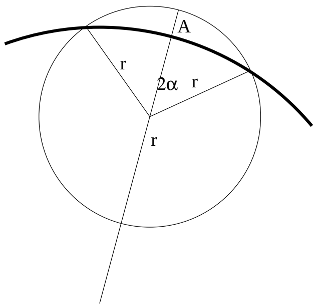
A simple trigonometric formula for triangles in Euclidean space is $A \sin\beta = B \sin\alpha$ which follows directly from expressing the height onto C in two different ways. Is there anything similar in the spherical case? We would never have been able to guess the version of the Pythagorean theorem in the spherical case from its Euclidean formulation. Formulas are formal and given a formula it is hence tempting to formally manipulating the structure, this is after all the point and beauty of a formula. What about $\sin A \sin\beta = \sin B \sin\alpha$? Why sine and not cosine? To get more symmetry! To make the sides and the angles be on the same footing! After all it is but a hypothesis, and let us see what it can lead to. If some absurdities it is obviously false and should be discarded. If A, B are small it reduces to the Euclidean case, which is good. If $\gamma = \frac{\pi}{2}$ we will get $\sin\alpha = \frac{\sin A}{\sin C}$ this is suggestive and one may be tempted to conclude that sines of lengths perform as lengths in Euclidan space. Considerations such this does of course not prove the formula, but it goes some way in explaining why it should be true (if it is true) and do indicate that something deeper, unclear what, is going on. This confirms the belief that mathematics is about discovery. That things are forced upon us. The properties of a sphere are like physical things, not up to our discretion but 'out there'. Now it turns out that the formula is true, and its truth can be verified by Euclidan geometry.

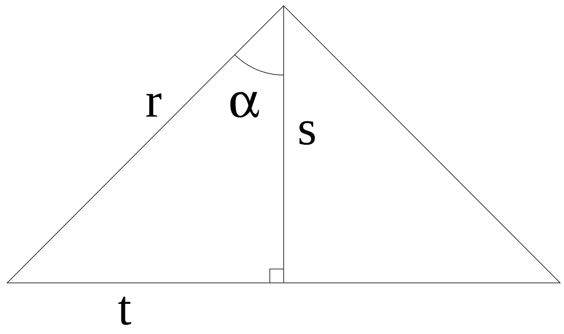
The Greeks knew about this as did their successors, including the Arabs, who did astronomy. With some simple formulas you can go a long way and in principle solve all the typical problems. As an example let us consider the following stronomical problem.

> An observer at latitude L^o ($L > 0$ on the Northern hemisphere, otherwise negatve) finds that a celestial body (a star, the Moon, or maybe even the Sun) culminates at a certain time T of the day and is then A^o above the horizon. Determine when it rose and when it will set.
>
> This is a problem the ancients could solve but which would baffle most people nowadays. As a mathematician you should be able to solve it using first principles and thinking of it three-dimensionally. It is not entirely trivial, but not that hard either, but the point of this exercise is to translate it into a problem involving spherical trigonometry.



The situation is as in the figure to the left. We have a horizon (H), making up one great circle on the celestial sphere, and the celestial equator (E) making up another. The center of the horizon is what is called by an Arabic word - zenith - the point just above you, its antipode (an alternative center) is called by another Arabic word - nadir. The center of the celestial equator will either be the Northpole N or the Southpole S. On the Northern hemisphere, the northpole is above the horizon, on the Southern hemisphere the southpole is above. Those two poles define the axis along wich the celestial sphere rotates, which means that an object will typically be above the horizon for some of the time, and below it for the rest of the time. At the poles, H and E coincide and every fixed object is either always above or below, while at the equator the poles lie on the horizon, and every object is above and below the horizon half the time. The complementary angle between H and E is denoted by L and coincides with the latitude of the observer. An object culminates when it is due south (on the northern hemisphere) and it will then also be at its maximal altitude [80]. The whole thing translates into determining the intersection between the small circle centered at S and with radius r intersecting the great circle H and to determine the angle 2α, as in the figure below.





Now from the observations we may figure out r. The angle E makes with the horizon H is $\pi/2 - L$, we thus get $r = \pi/2 - (\pi/2 - L - A) = L + A$ while $s = r - A = L$. We can now figure out t by $\cos(r) = \cos(s)\cos(t)$. Furthermore the sine-law gives $\sin(r)\sin(\alpha) = \sin(t)$. Some trigonometric manipulations eliminating t, which the Greeks would of course have ben capable of, yield $\sin(\alpha) = \frac{\sqrt{\cos^2 s - \cos^2 r}}{\sin(r)}$ which can be expressed in terms of L, A by e.g. $\frac{\sqrt{\frac{1}{2}(\cos(2L) - \cos(2(L+A))}}{\sin(L+A)}$[81]..

Spherical trigonometry was a basic component of the education of people trained in navigation, as it was part of the education of any astronomer until at least the sixties, while the computational powers of modern computers and calculators have made the shortcuts it provided for the manual computations obsolete. Nowadays few people have a command of it and it is definitely not part of the modern education of mathematicians.

Now it is important to realize that co-ordinates predates Descartes with a few thousand years, although in the version of spherical co-ordinates. Ancient people pin-pointed the positions of celestial objects in the sky, by declination and ascension. As the celestial sphere rotates, there will be two fixed points, one northern and one southern. For most of historical time there has been a fairly bright star close to the pole, the so called polar star[82] making things a bit simpler. Any two anti-podal points will define a unique great circle, the locus of all points equally distant to the two. This will be the (celestial) Equator. Furthermore we get a system of small concentric circles, the latitudes. Furthermore we will also have a system of great circles through the poles, which are the meridans. How to assign numbers to the latitudes is rather natural, namely the angular distance from the equator with sign. Sign is of course not necessary, as numbers are used for primarily descriptive purpose not computational, so classically one talks about northern and southern latitudes, and refer to them as declinations. There is also the notion of altitude as noted above, that refers to the angular distance to the horizon and changes during the day as the celestial sphere rotates, as we have considered above. One may make the remark that any object on the celestial equator spends equal time above and below the horizon. On the nothern hemisphere any object with a northern declination spends more time above than below, but the opposiye for objects of southern declinations. The so called fix-stars have fixed positions on the sky, while some objects such as the Moon and Sun and certain wandering stars, thus called planets, move over the year. The Sun follows a path which is a great circle and called the Ecliptic, and as it turnes out also the Moon and the planets do not stray far from it. The Ecliptic cuts the Equator at two antipodal points refered to as the equinoxes. When the Sun appears at the equinoxes day and night are equally long all over the Earth. One f them appears in the spring, the so called vernal equinox, the other in the fall, the so called autumnal. The meridan through the vernal equinox is set set as the zero meridan, thus the situation is somewhat different from the Earth, where the zero meridan is a question of convention. All celestial objects rise in the east and set in the west, this is a basic fact that should be known to all and sundry[83]. Thus an object further east will rise later (and set later), this accounts for the convention of dividing the celestial equator into 24 hours counted eastwards, i.e. clockwise. The corresponding time is referred to as the Right Ascension. If the Right Ascensions oftwo objects A and B are given as a and b respectively the time difference between their culminations will be given by $a - b$. The Right Ascension of the Sun increases durin the year. One reason that the Babylonians divided the cirle in 360^o may be that this is close to the number of days in a year. Thus the Sun moves roughly 1^o per day[84] which translates into 4m a day, which is the discrepancy between the time of rotation of the Celestial Sphere and the 'dygn'. Note thus that the Sun moves in the opposite direction of the rotation of the Sphere. Thus if the Right Ascension of the Sun is at $0h$ at the vernal equinox, it will be at $12h$ at the autumnal, and at $6h$ and $18h$ at the midsummer and midwinter[85].

Now in astronomy, there are always qualifications. Any statement is an approximation of a potentially indefinite ladder. The tilted Ecliptic appears because the axis of rotation of the Earth is tilted with repect to the orbit around the sun[86]. The axis of rotation is not fixed in space, in fact it describes a small circle around the normal to the Ecliptic with a radius of approximatively 23.5^o (which varies of course slightly) this is also the angle between the Equator and the Ecliptic. Now already the Greek discovered the precession of the equinoxes [87] amounting to about $50''$ a year, which would correspond to a period of some 26'000 years[88]. As a consequence stellar positions are updated every fifty years. To do so involved a lot of calculation, the spherical co-ordinates are not very well adapted to describe rotations, except of course to those around the axi. To do so in any reasonable manner you need to resort to the ambient Cartesian co-ordinates.

These kinds of observations have been made since the start of civilization. The Babylonians have very long records of celestial observations, we are talking about a thousand years or so, from which they could, in the spirit of statistics, find regularities and hence make predictions, without really understanding what was going on. The mathematics involved is of course not very advanced, but advaced enough that even today most people in the street would not beable to give coherent explanations of the movement of celestial bodies, and during the time it was the pregorative of a tiny elite both priestly and scientific at the same time. Astronomy is often seen as the earliest of sciences, and it certainly deserves the epiteth. It is noteworthy that apart from conventions (such as how to handle leap years) different cultures came up with very similar results, after all they shared a common reality.

Studying the heavens was always essential for finding your way around on the Earth, after all the Celestial sphere provides a common reference point, hence the importance for navigation which would have been impossible without it. Thus one may wonder how civilizations would have developed had the Earth been permanently under a thick cloud cover (which would not have prevented the emergence of life). For one thing the fact that the Earth is not flat would have taken longer to realize, local considerations are too crude, and the horizon would not have been an option in our permanent fog. The fact that the Earth is a sphere (at least approximately) is ancient knowledge and is more or less forced on you. Your latitude can immediately be read off from the way the celestial Equator is angled withthe Horizon, in practice by determining the altitude of the North Star [89] the longitudeis a bit more subtle but for stationary objects not that difficult, what is needed is to witness some celestial event which defines simultaneity at different locations, and then observing the differences in local times, as seen by the observed altitudes of the Sun. However, on a moving location, where determination has to be done in real time, and when observation of celestial phenomena can be difficult, the determination of longitude turned out to be a real challenge. It was eventually solved by methods going beyond astronomy, namely by stable and accurate clocks (but still of course involving some basic celestial observations such as determining the local time of the day). It is a general fact that as technology advances problems can be attacked in conceptually much more primitive ways.

Now through navigation the spherical co-ordinates were imposed on the Earth, and with that the problem of map making arouse. This means the problem of flattening the earth, of making maps on flat paper, much handier than the bulkiness of globes. This can of course not be done even on a local level, no matter how small a part, there will still be discrepancies, as we have already noted above. Theproblem really belongs to differential geometry and really the first serious problem which occured in the discipline, we will hence postpone to the appropriate section below.

Projective Geometry

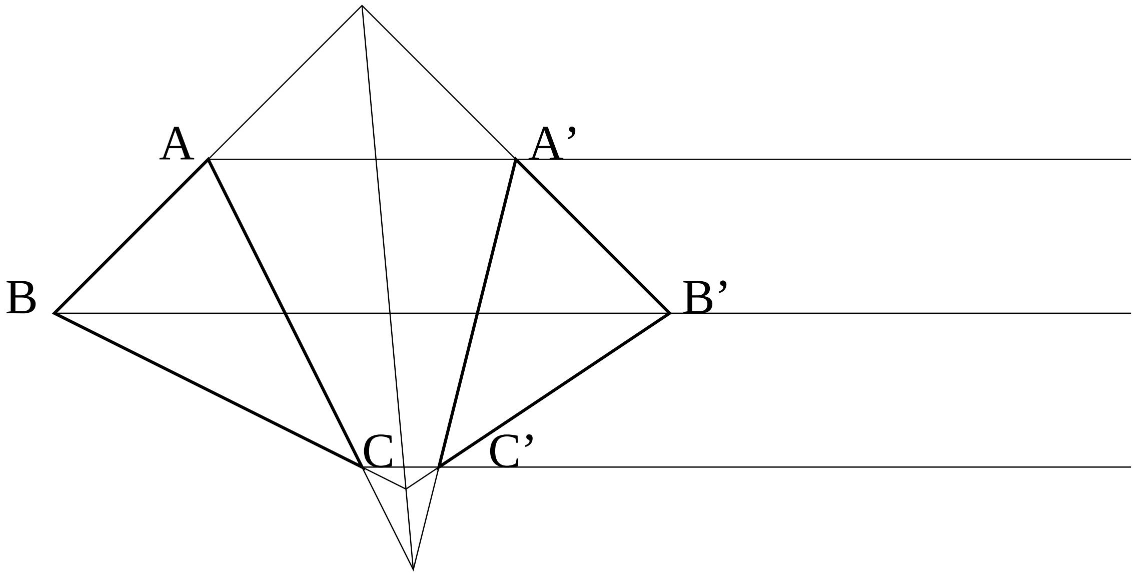
Projective geometry started already in the Hellenistic period with Pappus theorem. One may also think of Apollonius in this tradition, because projection plays a central role. The cones which figure and are cut by planes are not solid cones which are sawn through but typically lightcones (i.e. formed by a light source) which naturally fall onto various planes (walls, floors, ceilings), and less directly, the image of a circle from a slanted perspective is an ellipse. This is how we naturally see ellipses in nature[90]. Now the principle of perspective is such a simple one and one may be puzzled why it was not elucidated earlier in the history of mankind. Why did not the Egyptians employ it, or the Greeks? As to the latter they certainly were familiar with foreshortening as their decorations on vases testify to, bit that isa bit different and based more on observation than underlying principles. As to the Egyptians one should keep in mind what was the object of their pictoral representations, probably not mimesis. Their sculpture was realistic and sophisticated, but it may be easier to sculpture, at least in malleable material such as clay, then to paint on a surface. I do not know when the camera obscura was invented, it is such a simple invention that it would have been possible even in technologically primitive societies, but the point is what purposes would it have served. Inventions are made all the time but if they do not get any purchase they are quickly forgotten[91].

Now projective geometry is about points and lines and no metrics are involved. Shapes and sizes, i.e. angles and lengths are distorted, what is preserved are more abstract configurations, including coincidences and such things. Thus the group of transformations, to take a late 19th century perspective, is much bigger than those involved by rigid motions and scaling in Euclidean geometry. As noted in the previous lecture, projections from a point give rise to maps between two planes away from the point. Compositions of such maps are called perspectives. One important thing is that all conic sections are equivalent under perspectives, something implicit already in Apollonious.

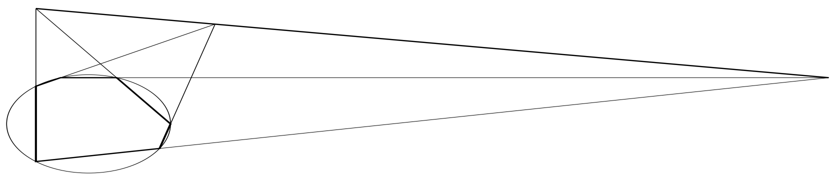
Now the interesting thing about projective geometry is that it will take place in the plane with a line at infinity added, where parallel lines meet. This line is not endowed with any physical existence but surely serves a mere convenience, as many concepts in mathematics. In particular it was never presented as a substitute for euclidean geometry, nor did the points at infinity cause any controversy, unlike the introduction of complex numbers, whose status for some time was uncertain.

Now projective geometry invites the concept of duality, as not only do two distinct point determine a line but two distinct lines determine a unique intersection point. Thus the formal roles played by lines and points are identical. This gives a first indication that the notion of space can be vastly extended by also being played by more extended geometrical concepts, such as the space of lines and conics etc. This is a development which took place in the 19th century, which also saw the establishment of projective geometry as an important discipline on its own, and was taken to extreme abstraction during the 20th century.

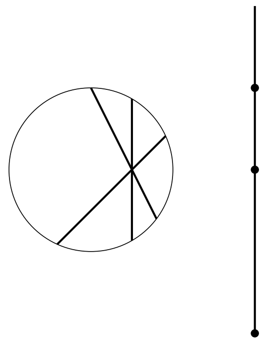
Pioneers of projective geometry in the modern age are Desargues, Pascal and La Hire. To Desargues we owe Desargues' theorem. Given two triangles ABC and $A'B'C'$ which are in perspective (cf figure below)



Then the intersection points of the three pairs $AB, A'B'; BC, B'C'; CA, C'A'$ lie on a line. This is immediate if the two triangles would lie in different planes, as the corresponding points lie in both, and hence on the intersection. Then it is natural to perturb the triangles, would they lie in the same plane, and derive the theorem by continuous movement, which then becomes a general principle in projective geometry somehow transcending the subject. Desargue proved a large number of facts through a systematic employment of perspectives and also dualities. Pascal as a sixteen year old discovered a celebrated theorem on hexagons inscribed in conics, namely that the intersections of opposing sides lie on a line.



It illustrates that not every hexagon can be inscribed in a conic. LA Hire finally developed systematically the ideas of Pascal and Desargues proving some 300 theorems, including those in Apollonius, but through some general methods he considered not only superior to the *ad hoc* methods of the Ancients but also to those employed by Descartes. Although the notion of polars was not explicated until the beginning of the 19th century by Gergonne La Hire proved theorems about them already in the 17th century. Through any point P outside a conic C are we able to draw two tangents, the line connecting those two tangent points is called the polar to P, conversely a line L intersects the conic in two points and the corresponding tangents meet in a point called the polar(point) of L. Thus any conic sets up a natural isomorphism between lines and points, i.e. between the plane and its dual[92]. La Hire proved that if we have a line L and Q a moving point on it, the corresponding polars will all go (rotate)through the polar point of L.



The projective plane as a metric entity is intimately connected to spherical geometry. The set of great circles of a sphere is a natural manifestation of a projective plane. As such it becomes one of the two non-Euclidean geometries. The distance between two great circles is naturally given by the angle they form. The smaller the closer they are. Two great circles intersecting at right angles make up for the most distant. Every great circle defines a pair of anti-posal points, and the minimal distance between two of them will be given by the angle of the two corresponding intersections. Thus we may think of pairs of anti-podal points as the points, and this amounts to identifying anti-podal points on a sphere. Thus $\mathbb{R}\mathbf{P^2} = \mathbf{S^2}/\sim$ with the induced metric. Great circles will descend to 'lines' intersecting always in one point, and there lengths will be π rather than 2π on the sphere, and the area as 2π rather than 4π while angles will be the same. Topologically what we get is a so called non-orientable surface, which cannot be embedded in our Euclidean 3-space [93]. If we look at the sphere we can consider a (narrow) symmetrical band around the equator. It will separate the sphere with two disjoint discs in its complement. Those discs are interchanged by the antipodal map, so one will do as representatives for the orbits. A cylinder under the anti-podal map will become a so called Moebius band or strip, with a connected edge and only one side. The projective plane is formed by gluing a disc along the boundary of a Moebius strip, which may sound simple enough but is, for reasons above, impossible to do[94].

Now to return to conic sections. A conic section which does not intersect the line at infinity will be an ellipse. Once it is tangent to it it turns into a parabola, and when it intersects it in two points, the tangents at those two points will constitute the asymptotes (lines tangent at infinity) and you have a hyperbola. When you cross the infinity line on the projective plane, you do not enter into a void, but return to the plane at an antipodal point (i.e. the same point). Thus a line in the real projective plane does not disconnect the space. Following one branch along across infinity you enter the other branch where the same asymptote intersects.

The notion of projective planes extends to any dimension, and in fact to every field of definition, in particular that of the complex numbers, which will sefve as the geometrical basis for algebraic geometry.

Co-ordinate Geometry

Descartes is known for his cartesian co-ordinates. First he is not the first who introduced co-ordinates in space, spherical co-ordinates have, as we have re-marked above, a very long pedigree. Nor did he present it in the standard way we are introduced to it today with two orthogonal axi (typically known as x and y axis, but in older literature abscissa and ordinate). Instead he considered any two lines or line segments and often only considered what we now would call the first quadrant, i.e. only using positive co-ordinates. This was of lesser importance. The main point was not to represent the plane by two numbers, which is as noted a much older idea, but to make those numbers work.

> Incidentally this procedure has lead to the notion of the number line, which although seen as a great pedagogical device also has the disadvantages that often accompany such. It has lead to a confusion between real numbers (quantities) and natural numbers, as if the latter were a natural subset of the former. The Greeks were not confused about it, on the other hand that prevented them from treating quantities formally as numbers. It is also used as a device to explain negative numbers which inevitably enter once we think of numbers as positions.

Descartes was engaged in finding a method to do geometry. A mathematical method is a kind of meta-mathematics. Rather than to rely on ad hoc solutions, you want to have a way of getting through to them automatically. This would greatly extend the range of mathematics and also make it accessible to dull minds who are capable of acquiring the skills of a methodology but without any deeper understanding nor any penetrating imagination. In short methodology is about technology, and in a modern society lots of people acquire the benefits of technology without having any understanding of how it works, nor having any desire to find out. Thus technology adds to the alienation of man, while making him more and more distant from the sources of his existence. The history of mathematics is really a survey of the technological changes which have been brought about, and the new vistas such open up: just as that is the course of an ideal mathematical education.

Thus co-ordinate geometry is not about a new geometry, as spherical opposed to plane Euclidean, but about a new way of approaching already well-known geometries. What is new will be the type of questions asked about it and the way of approaching and attacking them. It is in this sense the lasting contribution of Descartes should be viewed. For him it was only a part of a general methodology of science but it was in the field of mathematics it turned out to be most precise and ultimately more useful.

The first technological advance in mathematics was algebra. With algebra the fundamental notion of a formula was introduced. Formulas for most people seem to be mere recipies into which numerical values should be inserted in order for numerical values to be extracted. Although this maybe the original purpose of formulas, they soon took on a life on their own. This illustrates the principle that even if a methodology takes over by replacing thought, thinking and creativity are not abandoned, on the contrary, they are only asked to operate on the next level. Thus formulas are dynamic entities, their interest lie in the way they can be manipulated and be changed into other formulas. In this way there is a kind of chemistry of formulas.

Descartes method was a way of making algebra work seriously in geometry. Of being able to translate a geometrical problem into an algebraic, and sometimes the other way round. Geometry was still the basis of mathematics, and formal constructs such as algebra could often be justified by a translation into geometry.

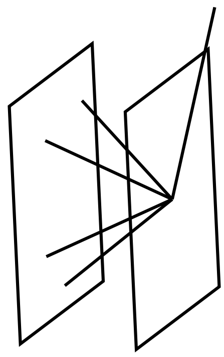
Solving geometrical problems with algebraic methods has become known as analytic geometry. It frees the mind from thinking, or at least to think at a geometrical level. It involves manipulations, less inspired by geometrical intuition than by symbolic contingencies. As a result even feeble minds can arrive at results which would have baffled the ancients. In a way Descartes is present at each such occasion and may feel (posthumosly) entitled to appropriate a large part of the credit. There are no patents in mathematics, so every mathematician is entitled to use the fruits of other people's labor for free.

In geometry it is very hard to visualize a geometric object in its generic aspects say a triangle. Each visualization will fasten on a specific one. This is not really a serious problem, as we have the power of abstraction, of only relying on the pertinent aspects of a specific example. On the other hand our powers are not unlimited, sometimes they err and from a figure we may draw unwarranted generalizations. This danger is very much reduced when you do algebra. The letters stand for numbers but you never need to specify the numbers in your mind, the letters, or better still the formal operations you make, see to it that you can entirely disregard them (occasionally you may not as a certain quantity may actually be zero, and then it should be handled with care, especially not be divided with, a source that lies behind many apparent paradoxes).

The new methodology not only solved classical problems, it also suggested new ones, which may never have bene thought of otherwise. It also may open entirely new vistas hitherto unsuspected. It was soon realized that lines were given by first degree equations, i.e. polynomials with only linear terms, while conic sections were given by second degree. This opened up for curves of any degree. Already in Hellinistic times the variety of curves for study had started to extend, some of them could be given algebraic equations, although not all of them, such as spirals. Anyway it lead to the discovery of curves which would have defied the imagination.

Cartesian co-ordinates are only one type of co-ordinates which could be imposed on a space. Polar co-ordinates were another alternative through which one was able to get simple expressions for curves (such as spirals) which had no algebraic counterpart. One cannot impose cartesian co-ordinates on a sphere, but polar work well, in fact the original spherical co-ordinates were polar co-ordinates. Curves such as cykloids, which would play an important role in the future, do not admit to any nice representations in either system but can be studied by the parametric representation. The idea of a method is not as rigid as a partiuclar manifestaion of it. The cartesian point of view could easily be modified and extended in any number of directions. Without the cartesian point of view it is hard to think that the methods of calculus would have been developed so quickly as they eventually did in the late 17th century. True graphic representations of velocities and distances existed already in the 14th century, Oresme is a name that comes to the mind, but without the idea that functions describing movements in space could be expressed as formulas further developments would have been stymied.

Co-ordinates can also be imposed on projective spaces. A line in the plane can be given by $Ax + By + C = 0$. The coefficients are only defined up to a multiple, i.e. the equation $\lambda Ax + \lambda By + \lambda C = 0$ defines the same line. However, it is important that one excludes $(0, 0, 0)$. Co-ordinates with that property are called homogenous co-ordinates. The space of all lines make up a projective plane, except for one line missing, namely the line at infinity. This can be formally written as $C = 0$ where $C \neq 0$! I.e. it is not satisfied by any 'finite' point. One may do the trick of introducing $\frac{x}{z}, \frac{y}{z}$. Plugging in and clearing denominators one gets $Ax + By + Cz = 0$ where (x, y, z) now are homogenous co-ordinates as the quotients $\frac{x}{z}, \frac{y}{z}$ are unchanged by multiples as above. We can then embed the ordinary plane $\mathbb{R}^2$ into the projective plane $\mathbb{R}\mathbf{P}^2$ via $(x, y) \mapsto (x, y, 1)$ and the missing line at infinit is given by $z = 0$. From the point of view of perspective, one can think of $\mathbb{R}\mathbf{P}^2$ as given by all the lines through the origin in $\mathbb{R}^3$. One may think of that as all the rays emenating from an artists eye, although a ray and its opposite will be identified. If we put a plane Π (canvas?) outside the eye P all the lines not lying in the plane Π' parallel to Π and passing through P will intersect Π. The one which do not, will make up a line - the line at infinity - and we see how we complement $\mathbb{R}^2$ into $\mathbb{R}\mathbf{P}^2$. The direction vectors (x, y, z) of the lines, will then serve as homogenous co-ordinates. Note that if we would use rays instead of lines, we would instead get the sphere S^2 and we see how the sphere is the double covering of the projective plane.



Given homogeneous co-ordinates it will be obvious how to extend the notion of the projective plane to any dimensions or to any field of definition, just as in the euclidean case, we get an immediate extension to any dimension, and field of defintion. Without co-ordinates that would have been very difficult. In fact cartesian co-ordinates have inspired the notion of cartesian products $A \times B$ of any two sets or structures A, B a fundamental constuction in mathematics.

Synthetic Geometry

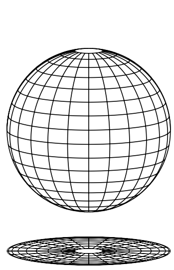
Synthetic geometry did not exist before co-ordinate geometry. Once again it is a question of method not content. Before Descartes all geometry was synthetic by default, then in the 19th century it became a reaction to the mindless algebraization of geometry. To prove things synthetically means to use your visual imagination and not to rely on calculations. The advantages are that the proofs are instructive and give you insight in what is really going on and ideally beautiful. The disadvantages are that there is no general method and the arguments tend to be ad hoc. Thus many of the results which may be routinely obtained by the cartesian method may be quite hard to achieve in synthetic ways. On the other hand there are problems that can only be solved synthetically and synthetic geometry tend to seek out different kinds of problem and is hence to be seen as a complement not a rival.

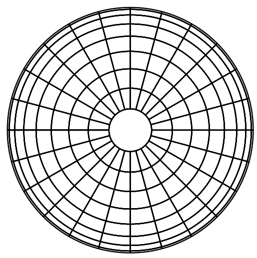
Hyperbolic Geometry

Hyperbolic Geometry was discovered already in the 18th century, the first century which saw serious and sustained efforts to prove the Parallel postulate. A favorite method of a mathematician employs in order to prove a fact is to assume that it is false and draw the consequences thereof. If they lead to a contradiction we conclude that its falsity is false and hence that it is correct. What has been done in effect is to create an imaginary world only to have it destroyed in the end, and out of its ashes the truth of the desired fact washes out, as so much gold panning out. Now there is one qualification to that kind of reasoning, the denial of so called the excluded middle, bit its discussion will be postponed to the final lecture. What the mathematicians of the 18th century did was to derive absurd consequences. But absurdity is not the same as contradiction, and they did not have the courage and faith to recognize what they discovered as real. Their absurdities would in retrospect become the new theorems of an entirely new and unexpected geometry. A more detailed question of which will be postponed to the lecture on Gauss. Gauss was the first who realized the possibility of the logical consistency of this kind of geometry, but was not the first to publish, that honor belongs to the provincial Russian Lobachevsky. Suffices it at this stage to point out some salient features. First of which instead of having an angular excess we have a deficit, the larger the area the larger the deficit. In fact with an appropriate normalization, as in the spherical case, there will be canonical units, that deficit is given by the area of the triangle. Thus there is an upper bound on the area of a triangle, those are given by triangles whose sides are pairwise parallel (but not all parallel to each othger!) because in that case the angles are all zero, and the vertices are placed at infinity. Furthermore the formulas for the circumeferences and areas of circles are the same as in the spherical case (with the same appropriate normalization), except that the usual trigonometric functions $\sin(x)\cos(x)$ are replaced by $\sinh(x), \cosh(x)$ functions which were 'discovered' and defined during the 18th century. Recalling the definitions we set $\cosh(x) = \frac{e^x+e^{-x}}{2}, \sinh(x) = \frac{e^x-e^{-x}}{2}$ those functions will satisfy addition laws which are very similar to the trigonometric ones as well as similar identities such as $\cosh^2(x) - \sinh^2(x) = 1$, and in fact there will be hyperbolic trigonometry simply by making the same substitutions as in the spherical case. In fact this was done by the French mathematician Lamberth before the acceptance of hyperbolic geometry. This is a tribute to the power of formulas and this was done very much in the formal spirit of that century. While the trigonometric functions are periodic and bounded, nothing like that holds for the hyperbolic, in fact they grow exponentially, $\cosh(x) \geq 1 = \cosh(0)$ and is even while $\sinh(x)$ is odd and growing[95]. This has some striking consequences. The areas and circumferences of circles grow exponentially with the radius (the same holds of course also for spheres) thus most of the area of a large circle is concentrated close to the circumference. There will also be parallax even for objects infinitely far away, as there will be right-angled triangles with two sides parallel defining a zero angle, and the remaining third strictly less than a right one. By the same token a straight line will not extend an angle of π in the visual field but will get smaller the further away one stands from the line[96]. Now just as in the case of Euclidean geometry, a sphere in hyperbolic geometry gives rise to spherical geometry. But in hyperbolic geometry the limit of circles passing through a fixed point and with the radius going to infinity is not a line but a so called horicycle, and in the same way spheres passing trhough a point and having greater and greater radii converge not to a plane but a so called hori-sphere which will be given the Euclidean geometry. The explanation for this will be given in a subsequent lecture.

Differential Geometry

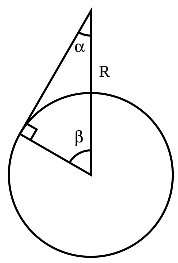
The natural way of depicting something in space (including the sphere) on a flat surface is by projection. The simplest one is so called orthographic projection, when the center of projection is infinitely far away, meaning that the rays are parallel[97]. It gives you the picture of the Earth as it would appear through a tele lens with infinite magnification infinitely far away. It is distortive. Lengths are not preserved, as we will understand they will never be so, nor are areas, angles, and great circles are usually mapped onto curves and not straight lines. Yet it does give a failry evocative picture, as all pictures of a 3-dimensional reality on a flat surface. We understand perspectival distortions and can make amends for it. The Earth, of course from now on assumed to be a perfect sphere, is mapped onto a circle, and if the center of projection is given by the lines parallel to the axis (i.e. the line joining the two poles) the meridans are mapped to lines through the pole and are equally spaced (the map actually preserves angles at the poles), while the latitudes are mapped onto concentric circles with the equator making up the boundary of the image. Such map projections are refered to as Azimuthal, and are codified by simply giving the spacing between latitudes, i.e. the radis $r(\psi)$ of the latitude given by ψ.





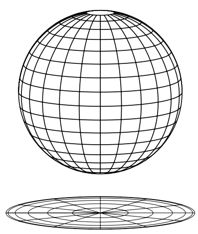
In the orthographic case $r(\psi) = \cos(\psi)$. A total area of 2π (a hemisphere) is mapped to an area of π (the image disc). Lengths along the latitudes are unchanged (constant scaling) while those along meridans change (scaling $\sin(\psi)$ the derivative of $r(\psi)$). This scaling also holds good for areas. Great circles are mapped onto arcs of ellipses, including straight lines (the image of the meridans).

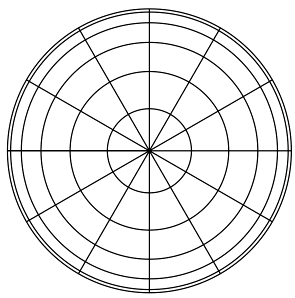
We may modify by putting the center of projection at some finite distance R, then a smaller part of the Earth will be visible on the other hand it will occupy a larger part of the visual sphere. The bounding circle, popularly known as the horizon, will no longer be a great circle (the equator if the projection point lies on the axis) but a smaller circle. It can all be easily worked out.

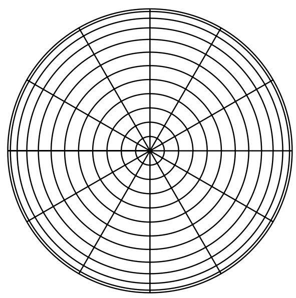


The angular extension α of the radius of the horizon in the visual field is clearly $\arcsin(1/R)$ while the angular extension on the earth is the supplementary $\beta = \pi/2 - \alpha$. It is an interesting question how small a circle on the visual sphere needs to be in order to be seen as curved, this easily translates into what height you must be avove the Earth in order to perceive that it is curved. If we project onto the plane perpendicular to the axis through the center we work out that $r(\psi) = \frac{R\cos(\psi)}{R-\sin(\psi)}$

We should keep in mind that this projection does two things. One projecting the visible part of the Earth, the other projecting the hidden part, which will be a mirror image as it will present the inside of the sphere.



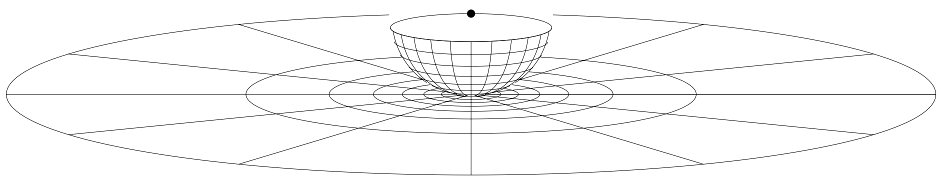


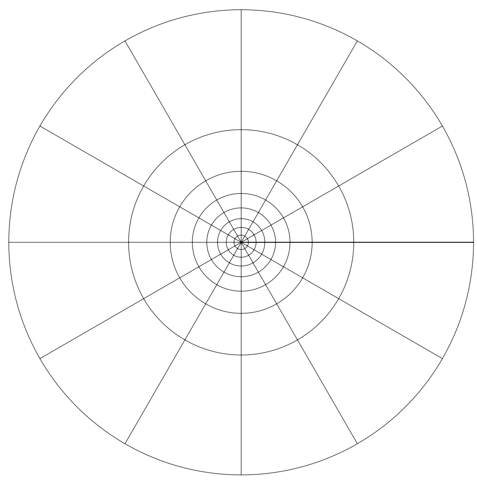


We note that the scaling of the latitudes will vary, the scaling factor will be $\frac{R}{r-\sin(\psi)}$, while the scaling factor in the orthogonal direction will be the derivative of $\frac{R\cos(\psi)}{R-\sin(\psi)}$ which is given by $\frac{R(1-\sin(\psi))}{(R-\sin(\psi))^2}$. If $R = 1$ both those scalings are equal, this means that the mapping preserves angles and is hence called conformal. The corresponding projection which maps the inside of the sphere minus the point of projection onto the entire plane is called the stereographic projection. It was known to the Greek and Hipparchos had shown by a synthetic geometric argument that it was indeed stereographic. The map r is then given by $\frac{\cos(\psi)}{1-\sin(\psi)}$.

Being stereographic means that the scale at each point is independent of the direction, thus locally they provide good approximations, the problem is that the scaling as such can vary from point to point. In particular there will be great distortions of lengths and areas. The scaling in the stereogrpahic case goes to infinity as we move away from the center.

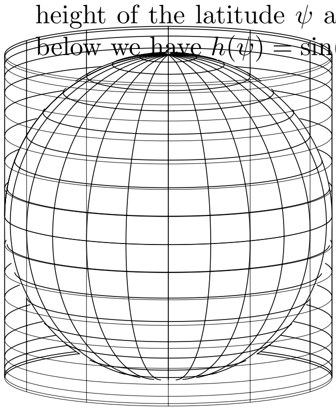
Finally if $R = \frac{1}{2}$ we have the so called gnomic projection, it projects one hemisphere onto the entire tangent plane at its center. If so clearly $r(\psi) = \cot(\psi)$. It also has the peculiar property that great circles are mapped onto straight lines, as those are given by planes through the projection point. Once we move away from the central point there will be local distortions no matter how small an area we look at. A gnomic projection can never be conformal, because then the angular sum of spherical triangles would be the same as that of planar ones. The scaling factor along latitudes will be given by $\frac{1}{\sin(\psi)}$ while the one along meridans will be the derivative of cotangent hence $\frac{1}{\sin^2(\psi)}$. This also gives a measure of the discrepancy from conformality, the further away from the center the more serious.

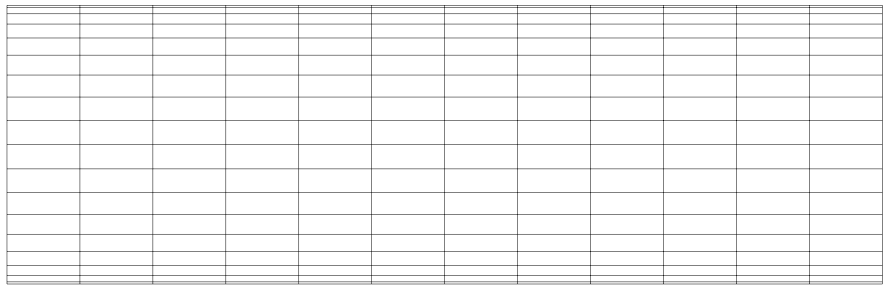




Now one may get any map projection of the Azimuthal type simply by varying the function r. If we want an area preserving one, we want that the latitudinal scaling offsets the one along the meridans. This gives rise to a differential equation. Namely $\frac{r}{\cos(\psi)}r' = -1$ [98]or $\frac{1}{2}(r^2)' = -\cos(\psi)$ which readily can be solved as $r = \sqrt{A - 2\sin(\psi)}$. As we want $r = 0$ for $\psi = \frac{\pi}{2}$ (the center at the Northpole) we obtain $r(\psi) = \sqrt{2}\sqrt{1-\sin(\psi)}$. This is known as Lamberts projection stemming from the 18th century, and it was not known by the Greek as it does not correspond to a natural projection.

A second type of projections are given by the cylindrical. Instead of projecting the sphere to a point, we now project it to a circumscribed cylinder. This was first done by Archimedes. The cylinders we can cut along a meridan and flatten it out. If this is done by a cylinder whose axis coincides with that of the earth, the longitudes will be mapped onto horizontal lines, and the meridans equispaced along vertical lines. Typically we will get a rectangle, but also an infinite strip. As above the projection is determined by a function $h(\psi)$ of the height of the latitude ψ above the central equator. In the case of Archimedes below we have $h(\psi) = \sin(\psi)$



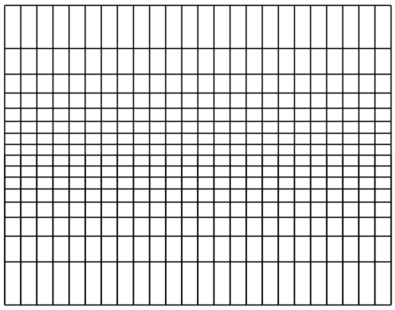


As all the latitudes have the same length the horizontal scaling will always be $1/\cos(\psi)$ at latitude ψ. The vertical scaling will be given by $h'(\psi) = \cos(\psi)$ we see indeed that areas are preserved, something that Archimedes already knew.

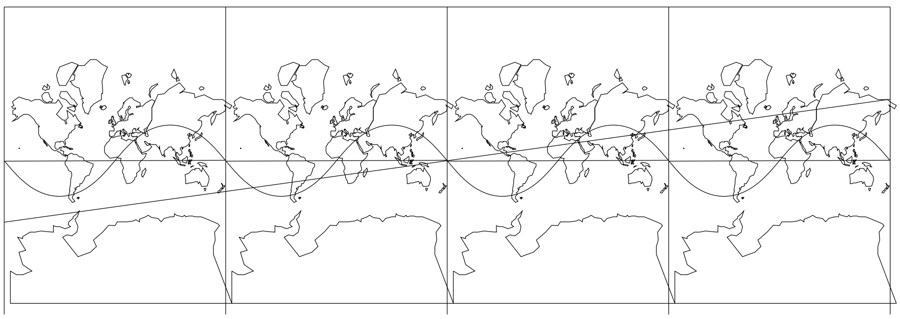
Is it possible to find a central projection which is conformal? This is not a question the Greek pondered. For us it amounts to finding a function h such that $h'(\psi) = 1/\cos(\psi)$. This was solved by Mercator in the 16th century and the solution was very important as it provided conformal maps very useful for navigation, because holding the course of constant compass direction corresponds to a straight line.

Now we can solve this by finding a primitive to $1/\cos(\psi)(= \sec(\psi))$ which can actually be done by elementary functions. The trick is to observe that $\frac{d}{d\psi}\sec(\psi) = \tan(\psi)\sec(\psi)$ and $\frac{d}{d\psi}\tan(\psi) = \sec^2(\psi)$ thus writing $\sec(\psi) = \frac{(\sec(\psi)+\tan(\psi))\sec(\psi)}{(\sec(\psi)+\tan(\psi))}$ we find a primitive as $\log(\sec(\psi)+\tan(\psi))(= \log(1+\sin(\psi)) - \log(\cos(\psi)))$. This is of course not how Mercator went about it. He was not interested in formulas, which would have been incomprehensible to him, and the relation between integration and derivation only became clear by Newton and Leibniz a hundred years later, but what he would have understood would be a numerical approximation of the integral, not unlike the way that Napier computed natural logarithms.

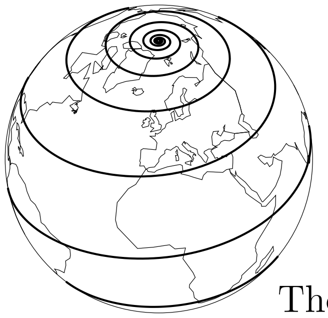
A typical Mercator grid may look like this



This straight line will not correspond to a great circle on the sphere unless it corresponds to a meridan or the equator, in general it will be a so called loxodrome which spirals from one pole to another. The reason for that is that first the rectangular picture is truncated at the poles. The distances between latitudes go to infinity, thus the entire sphere minus the poles go to an infinite cylinder. By slitting it along a meridan and flattening out, we are really considering the plane tesselated by an infinite number of strips as in the picture below.



The sinusoidal curve corresponds to a great circle, while the image of the straight line will be a spiral as below. It is an interesting exercise to figure out its length, in particular if it is finite.



The connection of conformal maps and the construction above to complex analysis will be explored in a a forthcoming section dealing with the rise of the latter.

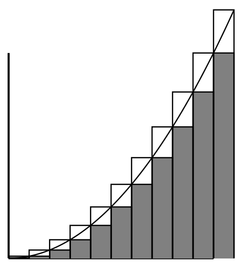
The precursors to Calculus

A classical problem is to compute areas of regions in the plane or volumes of spilds in space. A more subtle problem is to compute lengths of curves and surface areas in space. Everyone has a good intuitive notion of what is meant by area and volume, the basic idea being that if A is contained in B the area (volume) of A is less than that of B. In this way the notion of computing by exhaustion evolved already by the Greeks, and Archimedes was, as we have seen the master of it, proceeding in a strict rigorous way only to be matched at the end of the 19th century. What it amounted to was to set up a sequence of subareas or subvolumes X_n contained in a region or solid A whose areas and volumes could be computed in finite terms (as being made up of a finite set of things which could be computed and using the natural additive features of areas and volumes) in such a way that the left over areas of $A \setminus X_n$ would go to zero[99], hence the name 'exhaustion' meaning that we exhaust the idea. In the particular examples the ancients considered taking the limit was not a big deal.

The idea of computing lengths of curves is a very different thing. For one thing there is no simple and immediately obvious comparsion principle. Descartes along with his contemporaries even thought that it might not even be meaningful. The idea is that you somehow straighten out the curve, but how can you be sure that the length does not change, after all it is a pretty brutal thing to do to a curve. Another approach is to let a curve roll along a line, this only works if the curve encloses a convex regiom, or is part of it. But how do we know that it does not slip or slur? We may have a 1-1 correspondence between the points on the line and those of the curve, but this does not mean anything. We see that it is very tempting to look at those mathematical entities as being physical entities, that they have an almost material existence, and can be handled as physical objects. To decide what is the length of a curve is a philosophical question and its solution seems to hinge upon some conventions. Or more specifically, to talk about the length of a curve only makes sense in the context the question occurs naturally. If there is a reasonable context, the means ought to present themselves. Now when it comes to surface areas, the problems become even more subtle. We may convince ourselves that a curve can be straightened out, but can you flatten out a pice of surface? As we have seen you cannot even do it for a sphere, and we will discuss this more when it comes to Gauss.

As we noted, the Greeks, did some pretty impressive calculations of areas and volumes, especially Archimedes, which we have treated earlier in the book. And the theorem attributed to Pappus to the effect that the volume (surface area) of a solid of revolution around the axis of a line disjoint from a the rotated curve is to be expressed as the product of the enclosed area (length of) with the length of the circle traced by the corresponding center of gravities is also very impressive, connecting two things, ecah of which admittedly may be quite hard to compute.

Now during the early half of the 17th century when curves were given equations a more systematic way presented itself. Fermat and Cavalieri, the latter a student of Galileo, and many others managed to compute the areas under higher degrees polynomials such as $y = x^n$. In modern terminology $\int_0^a x^n dx = \frac{1}{n+1}a^{n+1}$. How did they do it? The case $n = 1$ is easy, we have just a triangle, but the case $n = 2$ of a parabola was a challenge to Archimedes.



Fermat proceeded as we would have done. He makes a lower and upper approximation of the area, the difference of which becomes smaller the narrower the rectangles are. In fact in the simple case of $b = 1$ we are looking at the sums

$$\frac{1}{N}(\sum_{i=0}^{N-1} \frac{i^n}{N^n}) < A < \frac{1}{N}(\sum_{i=1}^{N} \frac{i^n}{N^n})$$

Then one need to show the inequalities

$$\sum_{i=1}^{N-1} i^n < \frac{N^{n+1}}{n+1} < \sum_{i=1}^{N} i^n$$

and then everyting follows as it is easy to see that the difference between the two sums is given by the last rectangle, whose area can be made arbitrarily small, by choosing the subdivision fine enough, which is achieved by choosing N large enough. This is something that the Greeks could have done, although the proof of the inequality in general may have proved to be somewhat intractable for general n but not for low special values of n.

There is also another way of attacking it which could have been done at the time. For that we need some general scaling properties of the integral, which should be obvious by simply 'stretching'. Thus we have in modern notation

1) $\int_{ka}^{kb} f(x)dx = k\int_a^b f(kx)dx$
2) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
as well as the linearity of the integral
3) $\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
from which follows (by judicious choice of f, g) that
3') $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
We can now simply write, using 1) and 2)

$$\int_0^{\frac{1}{2}} x^2dx = \frac{1}{2}\int_0^1 (\frac{1}{2}x)^2 = \frac{1}{8}\int_0^1 x^2dx$$

furthermore

$$\int_{\frac{1}{2}}^1 x^2dx = \int_0^{\frac{1}{2}} (x+\frac{1}{2})^2dx = \int_0^{\frac{1}{2}} (x^2+x+\frac{1}{4})dx = \int_0^{\frac{1}{2}} x^2dx+\int_0^{\frac{1}{2}} xdx+\int_0^{\frac{1}{2}} \frac{1}{4}dx$$

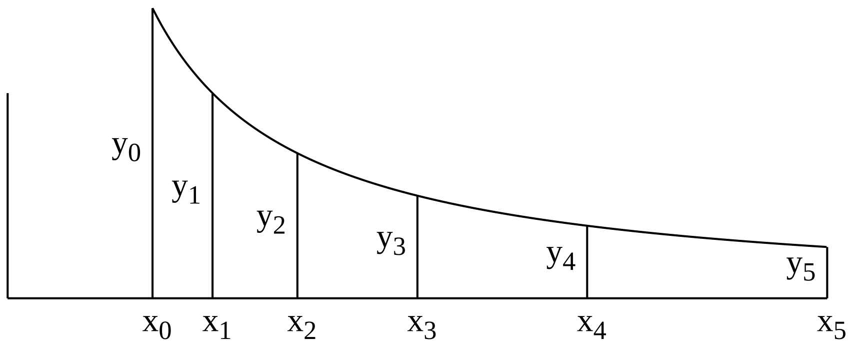
by principle 3'). Setting $V = \int_0^1 x^2dx$ and adding the two integrals on the left we end up with

$$V = (\frac{1}{8}V) + (\frac{1}{8}V + \frac{1}{4} + \frac{1}{8})$$

the two last integrals being elementary to evaluate. Solving for V we get $V = \frac{1}{3}$ directly.

The professional mathematician realises immediately that the method applies equally well to the any definite integral $\int_a^b x^2dx$ and that the inductive process allows the computation of any power x^n, and he encounters the challenge of what functions can be integrated by clever partitions and redistributions.

Fermat and others also tried to integrate arbitrary powers x^α where α could be negative and fractional but they encountered problems for $\alpha = -1$. It was noted that if we set (in modern notation) $L(\alpha) = \int_1^\alpha \frac{dx}{x}$ then $L(\alpha\beta) = L(\alpha) + L(\beta)$ from $\int_1^{\alpha\beta} \frac{dx}{x} = \int_1^\alpha \frac{dx}{x} + \int_\alpha^\alpha \frac{dx}{x}$ where for the last term $\int_\alpha^\alpha \frac{dx}{x} = \int_1^\beta \frac{d\alpha x}{\alpha x} = \int_1^\beta \frac{dx}{x}$ or just the scaling property of 1). Furthermore it was noted that this gave the natural logarithm, which then could be approximated, as Napier in fact had done, by approximating an integral. Another way of seeing it, as proposed by Gregory (1584-67)[100], is that if we choose points x_i such that the areas under the function $y = \frac{1}{x}$ and between $x_i, x_i + 1$ are constant, the corresponding y_i form a geometric progression (as do the x_i of course as well).



Those ideas were developed by Mercator of the eponymous projection. He studied the function $\frac{1}{1+x}$ dividing the interval $[0, x]$ into n equal parts. He then used as an approximation the sum

$$\frac{x}{n} + \frac{x}{n}\left(\frac{1}{1+\frac{x}{n}}\right) + \frac{x}{n}\left(\frac{1}{1+\frac{2x}{n}}\right) + \dots \frac{x}{n}\left(\frac{1}{1+\frac{(n-1)x}{n}}\right)$$

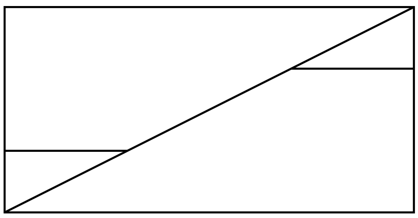
now each term $\frac{1}{1+\frac{x}{n}}$ can be written as a sum of a geometric series $\sum_{j=0}^{\infty}(-1)^j(\frac{kx}{n})^j$. We then get an infinite series, whose coefficients for x^k approximates the integrals $\int_0^1 x^{k-1}$ which when letting $n \to \infty$ equal them and hence we get the exact expression

$$(\log(1+x)) = x - \frac{x^2}{2} + \frac{x^3}{3} + (-1)^{j+1}\frac{x^j}{j} + \dots$$

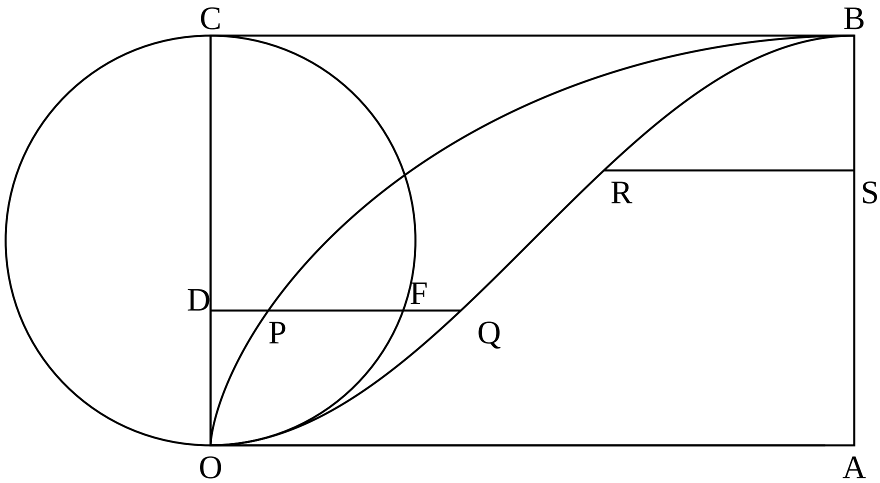
which could be used for numerical calculations[101].

Another more daring approach was done by Torricelli (1608-47), as noted a student of Galileo and associated with the barometer. By rotating the hyperbola $xy = k^2$ around the y-axis and considering it from $y = a$ to $y + \infty$. He managed to show that the volume was finite but the surface area infinite, which at the time was considered paradoxical[102].

One principle enounciated by Cavalieri (1598-47), another student of Galileo, was the so called method of indivisibles. An area was thought to made up of lines, if we had two figures for which we could make a $1 - 1$ correspondence between lines of equal length they would have the same area. The typical case is illustrated below, showing that the two triangles have the same area.



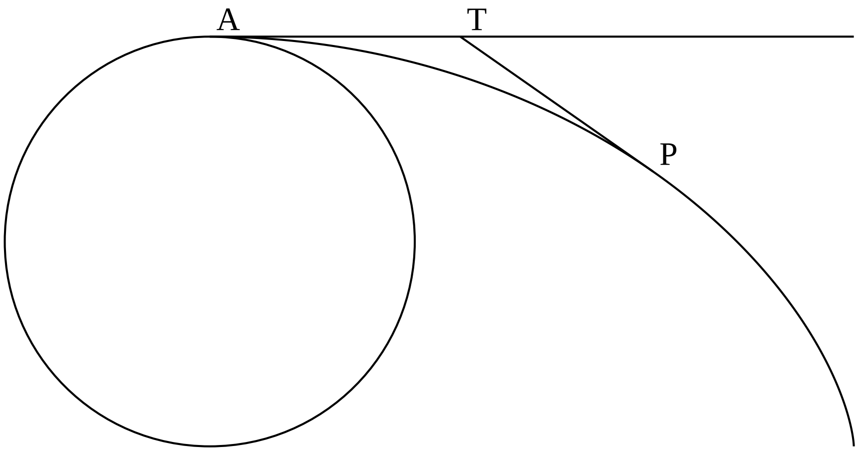
It is of course trivial to give counterexamples, so the method should be used with great care. On the other hand he is the originator of using cross sections (the salami method) to compute volumes of solids, based on the same principle with its obvious weaknesses. One non-trivial example is given by the French mathematician Roberval (1602-75)[103]. It concerns the problem of finding the area under a cycloid, the curve formed by a fixed point P of the perimeter of a rolling circle.



To each point P we associate a point Q such that the horizontal distance PQ is the same as DF. Now the point Q will describe an associated curve to the cycloid, and it will divide the rectangle $OABC$ in half, as symmetrical to DQ there will be a line RS of equal length. The area of the rectangle is twice as that of the circle ($=2\pi$ in the case of a unit circle) as its base is the length of the semi-circle, while its height is the diameter. The area bounded by the two curves wiill be the area of the semi-circle, because the lnegths PQ are equal to those of DF by construction. Hence the area under the cycloid is one and a half times that of the circle.

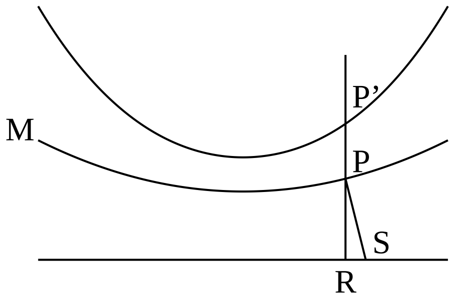
We note the great ingenuity of the approach and as the cycloid has a simple parametric representation it is not that easy to express it as a function y of x let alone finding a primitive. Note also that the method does not serve to compute the area under part of the cycloid, unless it is symmetric with repect to the vertical line that halves the rectangle.

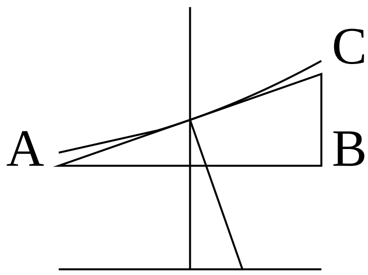
To compute lengths of curves, i.e. rectification, was tougher. Wren (1632-23) rectified the cycloid by showing that the length of the arc PA was twice that of PT where PT is tangent to the cycloid.



When it comes to finding tangents to curve, Fermat already had the right idea by taking chords and letting the two points coalesce and thus getting the limiting line. Given a formula for the curve, one could then form what later would be called the tangent. Fermat applied it to finding the extremal values of a function, by finding horizontal tangents. Descartes had a different method, he wanted to find circles tangent to the curve and hence find the normal. If the curve was given by $y = f(x)$ and we are interested in the normal to the point $(x_o, y_0(= f(x_0)))$ we want to find out when the circle $(x-u)^2 + (y-v)^2 = r^2$ intersects the curve $y = f(x)$ doubly. Now there will be many circles, all of whose centers lie on the normal, so we can assume that $v = 0$, furthermore in that case $(x_0 - u)^2 + f(x_0)^2 = r^2$ thus plugging in $y = f(x)$ we are interested in when $(x-u)^2 + f^2(x) = (x_0 - u)^2 + f^2(x_0)$ has a double root $x = x_0$ which will gives us a condition on u and the line from $(u, 0)$ to $(x_0, f(x_0)$. In practice the calculations necessary are unduly complicated[104].

Now Descartes' method of finding a normal to a curve was taken up by Van Heureat (1633-60?)[105]. He indicated a general method of reducing the rectification of a curve to finding the area under a related curve.





Given an arc MN we define an auxiliary arc by to every point P on the arc to associate P' such that $P'R$ is a (fixed) constant (k) times the distance PS where PS is the normal to P. Then we note that the differential triangle ABC at P is similar to the triangle PRS. From this we conclude in modern notation that k times the length of the arc MN is given by $\int_a^b k\sqrt{1 + (\frac{dy}{dx})^2}dx$ which of course is the classical formula. It is quite another thing how to compute it in special circumstances. People were stymied at finding the arclengths of parabolic arcs, but he was able to handle the cuspidal cubic (or as it was called then the semi-cubical parabola) given by $y^2 = x^3$ and other related curves with a singularity at the origin such as $y^n = x^{n+1}$.

James Gregory (1638-75)[106], not to be confused with the other Gregory. He took van Heureat's work further and asked whether any curve could occur as an auxiliary one to the problem of finding arclength. In practice it meant to find a function whose derivative was given. More importantly though it introduced the point of view that one should not just consider an area of a given region, but as a varying one, in the modern point of view varying the interval of integration. Isaac Barrow (1630-77)[107] was the first to actually state the fundamental theorem of calculus without really fully appreciating its significance, that was to be left to Newton and Leibniz.

Notes

[1]In many walks of life there are numbers applied, but much of that turns out to be just idle ornamentation, because only when the computational manipulations of those numbers make sense and reveal hitherto unsuspected things, can we talk about quantification in any true sense. Any other use is just peacockery and pseudo-science, because numbers are only numbers when computed with.

[2]More generally Newton also covered the case with varying mass and talked about the derivative of the momentum, which is mass times velocity.

[3]In classical language one speaks about heavy and inert mass, the latter related to Newton's law above, the former to the gravitational force it exerts. It is not clear that they should be equal, but they are considered to be. One of the questions whose significance was recognized by Einstein, and which he pondered.

[4]At least H.L. (1570-1619) a German-Dutch tried to patent his invention, it is possible that he was anticipated. The first telescope was a so called refractor based on two convex lenses with an upside down image, the latter could easily be reversed by adding an extra concave lens, but for astronomical purposes there is nothing to be gained by it and a lot to be lost.

[5]Galileo fathered three daughters out of wedlock. They would for that reason not be marriageable unless bribed with excessive dowries. Thus he had them choose careers as nuns. This gives an indication of social realities at the time not to be ignored, Only one of his daughters survived him.

[6]The four Galilean satellites - Io, Europe, Ganymede and Callisto - as they are nowadays called move in almost circular orbits, which accounts for the regularity. Discrepancies in the prediction of the clock, due to what is for all intents and purposes a 'Doppler' effect was later that century exploited by the Dane Rømer to determine the velocity of light, a task which Galileo had tried and failed at.

[7]This is a fascinating story. The British admiralty offered a prize for a reliable method of finding the longitude at sea.

[8]Recall that Galileo also discovered and studied sun spots by projecting the image presented by the telescope on a screen. Even the sun has its spots, as the saying goes.

[9]Galileo as well as Kepler were involved in astrology. A man after all needs to make a living and there has always been a premium on applications.

[10]That is a clever way of measuring the extension. A piece of hair is about $5 \cdot 10^{-5}$m thick. At a distance of 10 m it extends one second of arc. Due to the difficulty of accurately measuring such thin objects at the time, thicker strings were no doubt chosen necessitating removals at much larger distances, generating new problems. One may note that a star of first magnitude and an extension of $5''$ would if blown up to a disk the size of the sun have a magnitude of about -15 i.e. only ten times brighter than the Full Moon thus being a pale sun indeed. Note that brightness per area is independent of distance. The simple computation could easily have been done at the time, but I know of no instance of it. Furthermore absence of parallax gives a lower bound in terms of astronomical units of the distances to the stars. A parsec is defined as the distance at which the orbit of the Earth would extend one second of arc. A star at that distance, extending five seconds of arc would reach almost to Jupiter. Such big stars exist but not in the vicinity of the Earth.

[11]The comet was studied by Tycho Brahe who proved that it moved at a distance beyond the Moon by comparing his observation at his observatory at Ven with a con temporal observation down in Prague indicating a marked parallax of the Moon, but none perceived of the comet. Brahe made precise observations of its movement which has enabled posterity to

make reasonable guesses about its orbit, which turns out to be very eccentric, maybe even hyperbolic? At present it is estimated to be at a distance of 320 A.U. from the Sun (that of Pluto is approximately 40 A.U.). This was very important contradicting Galileo's claims that comets were nothing but optical phenomena in the atmosphere.

[12]The seminar was founded in 1556 intended for gifted students. Kepler must be considered the most distinguished alumnus, later on there were literary ones such as Hölderin, Mörike and Hermann Hesse.

[13]Kepler was also an accomplished astrologer adept at writing horoscopes, something we now may find as an embarrassment. But then as now there were strong pressures to do applications and traditionally astronomy had been closely linked with astrology. Anyway it provided a source of income and Kepler was often in dire financial straits.

[14]Descartes made a clear distinction between animals and men. The former had no souls and they could be explained in mechanical terms, they were mere machines. Humans had souls and partook of a dual reality involving matter and mind, whose precise interaction remained (and remains?) a mystery.

[15](1551-00)Also known as Bär, hence the Latin version of his name. He had been discovered as a herder of pigs in his late teens and been given an education and eventually becoming an Imperial astronomer and mathematician to Rudolf II, a position he later had to cede to Brahe. He is known for an alternate astronomical system close to the hybrid proposed by Tycho Brahe which caused accusations of plagiarism. Kepler made contact with him through a fulsome letter, which Bär had published in the preface to his book, something that did not initially exactly endear Kepler to Brahe.

[16]The former king Frederick II had supported Brahe by giving him the island of Ven and the funds to erect an observatory Uraniborg.

[17]One theory is that Brahe suffered from the ruptured bladder as a result of excessive politeness at a banquet. If true it must indicate a diseased one, as a vessel bursts at its weakest point, which in the case of a normal bladder is the entrance to the urinary tract.

[18]Kepler as an imperial employee enjoying the trust of the emperor was entitled to an ample salary, but payments were not always forthcoming, a Renaissance king was expected to run a profligate court and cash flows were often obstructed.

[19]There is a striking geometrical explanation of that based on the principle that the amount of light passing through a surface enclosed a source is independent of the surface. Thus concentric spheres around a source receive the same amount of light regardless of distance, as their areas increase by the square of the distance, the intensity falls of by the inverse square.

[20]This is puzzling at first. Why do we not experience the world upside down? Some reflection reveals that this does not make sense at all. Kepler however attributed the rectification to the workings of the soul.

[21]Apollonius wrote eight books, four of them extant in the original Greek, three through Arabic translations, and the last is lost. The first Western edition appeared in Venice in 1527, it would be followed by others, the most noteworthy being that of Halley in 1710.

[22]This is a very natural misconception in view of the 'curvature' decreasing the further we remove ourselves fro the apex. I recall entertaining it myself when being first introduced to conic sections as an emergent adolescent.

[23]It is not clear whether Kepler actually thought of the line at infinity, or rather the plane at infinity, having any physical meaning. If so he would conceive of the Universe as being real projective 3-space.

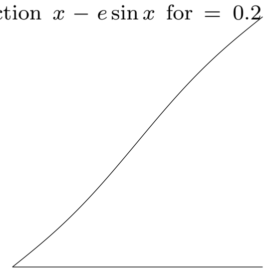
[24]The publication was delayed for many years due to legal troubles connected to his use of Brahe's data.

[25]Had Kepler turned his attention to Venus indeed, it is doubtful that he would have discovered his first law. On the other hand it was precisely because of its pronounced eccentricity of the Mars orbit which made it problematic and called for attention.

[26]Here is a list

Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune	(Pluto)
0.20563	0.00677	01671	0.0934	0.0485	0.05555	0.04638	0.00946	0.2488

[27]Here we graph the function $x - e \sin x$ for $= 0.2$ corresponding to the eccentricity of Mercury



On the y-axis one has a uniform spacing, lifting it back to the x-axis, will give a non-uniform spacing giving the variable speed. The angular variable A is referred to as the eccentric anomaly. Mote that the slope is steepest at the end points (perihelion) and lowest in the middle (aphelion).

[28]Some indication will be given by the luminosities, but those are complicated by phases, and anyway cannot be measured with desired accuracy.

[29]This explains the interest in Venus transits which occur in pairs every century or so, because they might give means of direct parallax to Venus.

[30]The hexagonal packing of circles in the plane is the most efficient. For space you look at layers of hexagonal packings of spheres and then add them by obvious translations, the way fruit sellers display their oranges.

[31]According to legend, Kepler considered eleven potential matches in succession and in the end settled for number five.

[32]The foremost general of the Catholic League, although he too was Lutheran. As a successful general he acquired a lot of land and power and was at the time of Kepler's death removed. Due to the defeat of the Imperial forces under Tilly against Gustavus II Dolphus at Breitenfeld in 1631, he was re instituted, held his own at the next engagement at Lützen in 1632 but was a few years later the victim of a plot and was killed.

[33]I measured the skies, now the shadows I measure/Skybound was the mind, earthbound the body rests

[34]Descartes also managed to arrange a comfortable situation through the investments in bonds in his late twenties.

[35]This is what you are told by Bell, to whom we will have occasion to return, other sources puts it to a year later, when a fourth child was to be born, and whose life likewise expired. It does not make too much of a difference. Descartes remained the youngest child and had no remembrance of his mother.

[36]There is of course a persistent opinion that men (and women) of genius seldom did well at

school, this is patently false, and more an expression of resentful jealousy than sober reflection.

[37]Descartes would always hold Jesuits in high regard testifying to his good memories of his early schooling. This would lead to strain relations with Pascal, who saw them as enemies. Relations may have been strained anyway.

[38]Eric Temple Bell's 'Men of Mathematics' does not contain much mathematics and can be said to be a book written for children or at least for very young adults. I myself read it at fourteen and my imagination was very much fired up, as early reading not seldom does to you as it tends to open up hitherto unsuspected worlds, and it gave me intellectual role models and as a consequence had a decisive influence in my life. Coming to those sketches in later life I am being made well aware of their obvious short-comings, but those were irrelevant at the time, after all in normal circumstances an individual grows and matures over time.

[39]The time has later been described by him as to abandoning study in order to meet all kinds of people and subjecting himself to all kinds of experiences with the goal of profiting from them.

[40]There is a story that while stationed in Breda, he came across a posting of a mathematical problem, which he solved and thus discovered within himself a talent for mathematical reasoning, and thus in earnest set him on the mathematical path.

[41]Dieudonné railed against the terminology disdainful of the central place it still occupied at the mathematical education at school into the middle of the 20th century, claiming that it should be reserved for the kind of geometry espoused by Serre in his article 'Geometrie Algebrique et Analytique'.

[42]Jesuits have still a rather bad name in traditionally Protestant countries, such as Sweden. They are rather thought of as devils incarnate. In fact their order was founded by Ignatius of Loyola as a reaction against Protestantism, and as such they formed a rather uncorrupted counterpart in their radicalization of thought and combativeness in their belligerency of purpose. This did not, for the same reason as already alluded to above, prevent them from playing an important role in the furthering of knowledge in the non-theological sphere as well.

[43]Like most satirists Voltaire missed the point, but that of course did not make his satire less enjoyable. The notion of a variety of possible worlds, or universa, has become quite popular in modern cosmology (with commercial success as well, at least to its popularizers), and the initial aim of the string theorist was to repeat Descartes performance, but with a vastly improved background and mathematical sophistication.

[44]Ostensibly by being shown the implements of torture. Galileo was a man cursed with imagination and needed no practical demonstration on his own body to get the point. Much as what goes for as physical courage seems to be founded on an inability to imagine. Physical courage ultimately means an indifference to physical pain, while moral courage primarily means an indifference to social ostracism. The two are distinct, but may overlap. It is possible that the latter is much more unusual than the former, especially if that is based on a failure of imagination. Galileo certainly had moral courage.

[45]Elizabeth (of the Palatinate) was born late in 1618 and died early in 1680, while Queen Christina was born late 1626 and died 1689, she was thus not a mere teenager at 19 when Descartes arrived, as Bell ever cavalier with dates and ages claims.

[46]He was about to go for a long time after his death. At first his remains were interred at a graveyard for orphaned children, (and would later be the site for the church of Adolf Fredrik built in the next century where a commemorative plaque has been installed). Then in 1666 the remains were moved to France, and after the French Revolution it was decided that he would be buried in the Pantheon, but that came to naught. In the end he was transferred to the Abbey at Saint-German-des-Prés in 1819, but still a finger is not accounted for, while his

skull is on display in the Musee de l'Homme in Paris.

[47]Mersenne pops up repeatedly in the mathematics of early 17th century France for reasons just alluded to. Here we have a very intelligent and able man who made no impact on mathematics save rather trivial ones. This is the lot for most of us, no matter how clever, educated and hardworking. It takes more to make a difference. In fairness Mersenne had wide interests and (pure) mathematics was just one of them. He was among many things interested in music and suggested the number $\sqrt[4]{\frac{2}{3-\sqrt{2}}}$ as a constructible approximation to $2^{\frac{1}{12}}$ in order to get a well-tempered scale. Mersenne is mathematically now known for so called Mersenne primes, primes of the form $M_p = 2^p - 1$ where p is a prime (it is easy to see that $a^p - 1$ is a prime only if $a = 2$ and p is a prime). Those are convenient to check for primality, and thus world-records for large primes are usually of that form. With Mersenne Fermat corresponded on pedestrian aspects of numbers such as factorization. Fermat suggested that a way of finding large divisors of a number (meaning close to the square root of it) one may add squares to it until one encounters a square. This means making it the difference between two squares and hence automatically factorized (in extreme cases the factors will turn out to be trivial such as $5 = 3^2 - 2^2$). Fermat had also observed that the numbers, now denoted by F_n given by $2^{2^n} + 1$ are primes for $n = 1, 2, 3, 4$ and boldly conjectured that they are all primes. This was wishful thinking of course, it fails for $n = 5$ as $2^7 \cdot 5 \equiv -1(641)$ and $5^4 \equiv -2^4(641)$ and thus $2^{32} \equiv -(2^7 \cdot 5)^4 \equiv -1(641)$. Easy to verify but why come up with 641? No more Fermat primes have been found than the classical. This is obviously the outcome of a search for explicit formulas for primes allowing you to write down arbitrarily large primes. Would p prime imply $2^p - 1$ prime one could easily write up huge primes such as $2^{2^{2^{2^3-1}}} - 1$ (incidentally Catalan suggested looking at the sequence $2, M(2), M(M(2)) \ldots$ which are primes up to $M_{127} = M(M(M(M(2))))$ as proved by Lucas in 1876. To go further and test for primality is no longer feasible). The Fermat numbers seem to turn out to be a typical dead-end. But they would unexpectedly turn up in the work of Gauss on constructions of regular polygons with ruler and compass more than 150 years later.

[48]yet of course publishers had found it worthwhile to have Latin translations made of it to be sold.

[49]Bell in his sketch of Fermat challenges any mathematically innocent to find a proof on first principles within say a year. He speculates, on whatever grounds, that of a million candidates at most a dozen would succeed. Thus for a young fledgling mathematician it can serve as an intelligence test and decide whether you have what it takes.

[50]It is in fact the starting point of anything interesting that is done on primes, and provides a convenient test for proving that certain numbers have factors without exhibiting any. It also proves to be an invaluable tool in secure codes, but that is another story emerging in the second half of the 20th century. There is of course an extension of Fermat due to Euler, but that is an immediate consequence of the idea that proves Fermat's version. There is much pedagogical value of presenting an idea in its simple form and not have it obscured by irrelevant, and ultimately rather mundane, technicalities.

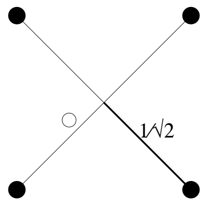
[51]Which would later have striking applications to coding by providing generalizations of Fermat's little theorem and the ideas involved. But that would only be explicated in the 19th century.

[52]Numbers of the form $4n + 3$ can never be the sum of two squares, as the residues of those mod four are $0, 1$ and hence for a sum $0, 1, 2$

[53]Note that we may replace b by $-b$ and get a new representation $(ac + bd)^2 + (ad - bc)^2$

[54]The principle is explained by Fermat in a letter, but the actual application to his statement has been lost.

[55]A modern proof would go something like this. Consider the Gaussian integers consisting of $a + bi$ with $a, b \in \mathbb{Z}$. First we need to establish the Euclidean algorithm for Gaussian integers, more precisely given P, Q there is an R such that $\mathrm{Nm}(P - RQ) < \mathrm{Nm}(Q)$ where $\mathrm{Nm}(a + bi) = a^2 + b^2$, The point is to find an integral approximation of the rational number $r = P/Q$



Such that $\mathrm{Nm}(R - r) \leq \frac{1}{2} < 1$. We then get $P - RQ = P - rQ - (R-r)Q = (r-R)Q$ from which we get $\mathrm{Nm}(P-RQ) < \mathrm{Nm}(Q)$ and we are done. But this is obvious from the picture on the left as the distance of any point to the closest Gaussian integer is clearly less than $\sqrt{2}/2$. Given the division algorithm we can proceed exactly as in Euclid and get unique factorization as with the integers. Now if $p|a^2 + b^2$ then either $p|a, p|b$ and hence $p^2|a^2 + b^2$ or -1 is a quadratic residue mod p which means that $p|x^2 + 1$ for some x simply by considering a/b or b/a depending on which of a, b is non-zero mod p. That $p \equiv 1(4)$ is equivalent with -1 a quadratic residue is somewhat harder to prove. One can show that there is a primitive element ϵ, such that $\epsilon, \epsilon^2 \dots \epsilon^{p-1} = 1$ make up all the non-zero residues. Note that the quadratic residues are exactly those for which ϵ are raised to an even power. As the equation $x^2 = 1$ has only two roots ± 1 ($x^2 - 1 = (x-1)(x+1)$ thus if there is a third root ζ the product of two non-zero residues $\zeta + 1, \zeta - 1$ would be zero which is absurd. the same idea can be used to show the existence of a primitive element, otherwise there would be an $m < n$ such that for all x we would have $x^m = 1$) we see that $-1 = \epsilon^{\frac{p-1}{2}}$ and thus a quadratic residue iff $\frac{p-1}{2}$ is even, i.e. $p = 1(4)$. Now in that case $p|x^2 + 1 = (x+i)(x-i)$. If p would be a prime we would have $p|x + i$ say but then also $p = \overline{p}|x - i$ and hence $p|x$ and thus $p|1$ i.e. a unit which is absurd as $\mathrm{Nm}(p) = p^2$ Thus p splits and we can write $p = \alpha\overline{\alpha}$ with $\mathrm{Nm}(\alpha) = p$ from which follows that p is the sum of two squares, in fact uniquely so, as any non-trivial factor of p must have norm p. This incidentally shows that all the primes of type $4n + 1$ (and $2 = (1+i)(1-i)$) split in the Gaussian integers (2 turns to a square modulo a unit) while those of order $4n + 3$ remain primes (such primes cannot be the sum of two squares).

[56]It has of course plenty of solutions for $n = 2$ known to the Greeks. They can in fact be given parametrically by $x = (s^2 - t^2), y = 2st, z = (s^2 + t^2)$ (with the role of x, y interchanged as well). It is trivial to verify that this is a solution, and only slighter more involved that it will provide all the solutions. For the latter we write $x^2 = (z + y)(z - y)$ and use unique factorization.

[57]In fact this was not the only marginal note found in Fermat's copy. It was his habit to jot down his commentaries and new results in the margins a convenient and commendable one.

[58]It is of course trivial to note that one may reduce to n being a prime or $n = 4$ the latter already settled by Fermat himself.

[59]This was a heretical movement within the Catholic Church founded by the Dutch Bishop Cornelius Jansen(1585-38) and led after his death by Duvergier and Arnauld with its center at Abbey de Port-Royale in Paris. Being known as Jansenism they stressed original sin, divine grace and predestination, claiming St-Augustine as a source. Thus they were of course very close to Calvinism. As a consequence they were bitterly opposed to the Jesuits. They incurred papal condemnation along with the prohibition of the teachings of St-Augustine. The sect survived nevertheless until the early 18th century when it split up into antagonistic factions.

[60]He is also known for Torricelli's law which states that the velocity of liquid leaking through a small hole is proportional to the square root of the height of the liquid, thus satisfying an equation given by $\frac{dy}{dt} = -k\sqrt{y}$ although of course this discovery was made before the invention of calculus and its notations, and would not be articulated in the modern way.

[61]pressure in many contexts are still measured in terms of mm Hg.

[62]This actually constituted a clever way of weighing the column of air delimited by the two points of measure.

[63]A language superseded by C which works on the same principles.

[64]So there is great flexibility as to how the game is played, one round could consist in the flipping of a coin, or both throwing dice in some ways to ensure a particular winner of probability one half. It is a game of pure chance meaning that any two gamblers will have equal probability of winning, never mind that we have not defined probability yet.

[65]He was a friend and collaborator with da Vinci and he is also remembered for being the first person to publish a book on double entry bookkeeping. His attitude to mathematics and the real social world was hence that of an accountant.

[66]In a game of 100 rounds and a lead of 10, one would use the ratio $\rho = 0.1$ as a basis, say by giving a share of $\frac{1+\rho}{2}$. But no matter what formula is used, the result would be the same for $99 - 89$ as for $10 - 0$ despite the fact that in the first case the leading player seems much closer to ultimate victory than in the second. Thus the solution suffers the same weakness as that of Pacioli.

[67]Looking at the case of $a = 1$ and $b = 3$ we have eight cases of three runs, namely $000, 001, 010, 011, 100, 101, 110111$ all of which are assumed to be equally possible, and where 1 means A wins, and 0 that B wins. Of those eight seven leads to win for A while only one 000 makes B win. In other words A is seven times as 'likely' to win than B. On the other hand not all of those eight runs would occur in practice, only $001, 01, 1, 000$ should we count those as equally 'likely' and thus get an answer of only three times as likely? If we only think of the runs as such with no interpretation the first one is the case, and in that background case we may sort out the relevant ones.

[68]This can be compressed to the idea of considering indefinite runs. We can then formulate very precise statements to the effect that if we have a fair coin and throw it N times out of the 2^N possible outcomes (assumed equally likely) those for which the frequency differs more than $k\sqrt{N}$ will make up a fraction of less than $\epsilon > 0$ provided N is big enough, where the latter obviously depends on ϵ and k. We are talking about Bernouilli's Law of Large numbers, the principle of which was already formulated by Cardano. The point is to somehow connect the mathematical notion of probability with that of 'real life'. A theorem above is really just a combinatorical exercise, although a far from trivial one. The connection with real life is a leap of faith, be it a rather reasonable one. Due to the asymptotic nature of such statements one can never reject any statement in any unqualified way, the experiment one does may after all be exceptional no matter how many trials are made.

[69]It is rare indeed that if you toss a fair coin a ten thousand times and there will be no sequence of ten heads in a row.

[70]Of course nothing new is introduced only that the assumption of Fermat is hidden. To return to a previous endnote, by Pascal's reasoning there should be as many outcomes starting with 1 as with 0 because are equally likely. If we would only count $1, 01, 001$ as leading to a win of A and 000 as a win of B there would only be one that starts with 1 but three which start with 0, in order to get an even balance we have to take into account the hidden series brought to attention by Fermat.

[71]$E(r,s) = \sum_{k=0}^{s-1} \binom{r+s-1}{k}$

[72]One may compare with the present fashion for finding evolutionary explanations for adoption of human values. There is something deeply unsettling and unsatisfactory about this.

[73]Of course the distance to the Sun and the Moon vary with time so in partuclar the notion of an astronomical unit will have to be specified, but this does not interfere with the principles

the discussion is concerned with.

[74]One may think of the smile of the Cheshire cat being seen without the face.

[75]It is thus symptomatic that when a definition of a meter was proposed as part of the great reform on units after the French Revolution it was defined as parts of a great circle, not as the there being 40 million meters to the equator but as 10^{-7} times the length of the meridan between the equator and a pole, the Earth being flattened, not all great circles are of equal length. (Note that on an ellipsoid one may still define great circles as intersections with planes through the center but only exceptionally will those be what would later be called geodesics. For a rotaional ellipsoids, the equator and the meridans will still have that property, the other intersections not). Now in practice one was reduced to producing a standard, an actual physical rod of platina of length one meter kept in a basement in Paris at a steady temperature, to more local protypes had to be calibrated. It is not so easy to measure the length of any extended arc on the Earth due to the uneven nature of its typography, and it is hard to find a meridan which only extends over oceans. But the advantages of an abstract definition should be obvious, because they can always, unlike concrete standards be duplicated.

[76]The Moon is fairly big on the sky, it is no mere dot. Had it been put on the Earth instead, 30 minutes of arc corresponds to about a circle of diameter 55 km. This is roughly the area occupied by London. Hence the area the Moon occupies in our visual field is roughly the same as that London occupies on the Earth! By the same token, the Moon could be drowned in the Swedish ;ake Vänern.

[77]The Earth has a radius of $6400 = 2^6 \cdot 10^2$ km. A circle with radius 1 km would then correspond to $r = 1/6400$. The error would be $r^2/6 = 2^{-13}3^{-1}10^{-4} \sim 1/24 \cdot 10^{-7}$ times the circumference. We are talking about $0.125 \cdot 10^{-7}$ km or 0.01 mm. The relativ error will grow as the square of the radius, and the absolute as the cube of the radius. In the case of a circle of radius 10 km, the error will still only be 1 cm, while for 100 km we will be talking about 10 m. Thus such discrepancies would not be noticeable on a local level.

[78]The combinatorial formula is a bit tricky to prove directly. The natural thing is to extend the notion to arbritary connected polygons with n edges and define the excess as $\sum \delta_i - (n-2)\pi$. If two such polygons are attached along edges, those need to form a connected path of k edges, otherwise the union will not be conected. If so there will be two end points and $k-1$ interior points. If we add all the angles, those at interior points will disappear and each subtract 2π from the sum. Thus if the angular sum is given by A_i the angular sum of the union will be given by $A_1 + A_2 - 2(k-1)\pi$. Thus the sum of the angular excesses will be $A_1 + A_2 - (n_1 + n_2 - 4)\pi$ while the angular excess of the union will be $A_1 + A_2 - 2(k-1)\pi - (n_1 + n_2 - 2k - 2)\pi$ and the two agree. This will allow us to use induction. In particular if a triangle is subdivided into triangles, we can split it up into two polygons by following a connected path of edges disconnecting the triangle, which incidentally will prove the combinatorial formula. The moral is that it is convenient to extend the notion of angular excess to include all polygons in order to apply an inductive strategy, you cannot express a general subdivision into triangles into a succession of simple ones (such as dividing a triangle by connecting a central point with each of the edges).

[79]Can we simplify the formula $< \alpha \times \gamma \cdot \beta \times \gamma >$? probably not, but recall that $< \alpha \times \beta \cdot \gamma >$ gives (up to a factor 1/6) the volume of the tetraheder spanned by α, β, γ. Thus the first scalar-product tells us that $\beta \times \gamma$ lies in the span of α, γ which is another way of expressing the orthogonality at γ. This means as $< \beta \times \gamma \cdot \beta > = < \beta \times \gamma \cdot \gamma > = 0$ that if $\beta \times \gamma = A\alpha + B\gamma$ (necessarily $A, B \neq 0$) that $A < \alpha \cdot \gamma > + B = 0$ as γ is a unit vector, and $A < \alpha \cdot \beta > + B < \gamma \cdot \beta > = 0$ that indeed $< \alpha \cdot \beta > = < \alpha \cdot \gamma >< \beta \cdot \gamma >$ by solving for $B = -A < \alpha \cdot \gamma >$ in the first. But does this give any explanationor is it just a manipulation, so common in mathematics?

[80]The sun ideally culminates at noon, but for a variety of reasons, if that would be imposed as a definition of noon, the length between two succesive culminations would vary, thus one

chooses to fix an average of culminations and calll it a 24 hour period ('dygn' in Scandinavian langages), while the period of rotation is much more stable and amounts to $23h56m$. The discrepancy of 4 minutes over a year make up 24 hours. The way the culminations of the sun oscilates around noon is decribed by the so called equation of time. The effective is cumulative and makes up at its maximu a quarter of an hour. If the Earth's axis would not tilt, and its orbit would be a circle, this phenomenon would not appear.

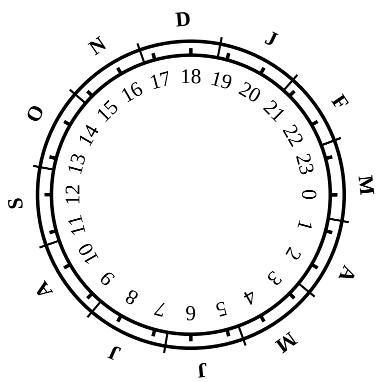
[81]The delination of the brightest star Sirius is -17^o thus it is invisible in the artic regions north of latitude 73^o. Thus in Northern regions such as Scandinavia it is not as prominent as it is further south. At the city of Gothenburg at 58^o it culminates at $(90-58)-17=15^o$ above the horizion, and stays above it for 3h40m. In fact plugging into the formula we get $\alpha = 27.52^o$ which easily is converted into hours, keeping in mind that 360^o corresponds to $24h$ (actually $23h56m$. Now for Sirius to be visible, this rather narrow window better occur when there is night, thus Sirius is only visible during the winter at such latitudes.

[82]Actually Polaris as it is called more officially is closer to the pole now than it was the case in classical time, but not for much longer.

[83]It seems so to be, after all people looking at apartments generally understand that if the balcony faces east they can expect morning sun, not evening sun. Furthermore on the East Coast of the States, the sun will rise above the Atlantic, but on the West Coast it will set in the Pacific. Nevertheless T.Hall in his biography of Gauss, noted how amused as well as annoyed the latter was when reading in Walter Scott that the Sun rose in the West, and he corrected it in all the copies he could lay his hands on.

[84]As already noted this is not constant. Even if the Sun would move at a uniform angular speed along the Ecliptic, it does not do so with respect to the divisions of the Ecliptic, given by the meridans, as the Ecliptic is tilted with respect to the Equator. An additional complication is that the Earth moves in a slightly elliptic orbit around the sun and hence due to Kepler's Second Law, its speed and hence the apparent one of the Sun will vary.

[85]The vernal equinox is around March 21, each year it is moved forward by a quarter day, thus each fourth year it is brought back a day by the adding of a leap day during the leap year. This is the principle behind the Julian calender. We may set up a rough conversion between the days of the year and the Right Ascension of the Sun as below.



Now the Right Ascension of Sirius is $6h45m$ this means that it will in the summer culminate about the same time as the Sun, this is not a good time to observe it. The best time is when the time difference is at its greatest, which means when the Sun is at $18h45m$ which translates into around New Year, eleven days past the midwinter. With a bit more care we can also work out exactly when during the year Sirius will be visible at night at given latitudes.

[86]Whether the Earth actually rotates or not, or whether the Earth orbits the Sun or vice versa does not matter. Mathematically we have two planes, one defined by the rotation of the Earth (or the celestial sphere), whose rotational axis is a normal to the first; the other by the relative movements of the Earth and Sun visavi each other.

[87]Incidentally by the Hellenistic astronomer. geographer and mathematician Hipparchus

($I\pi\pi\alpha\rho\chi o\zeta$) 190 -120 B.C. He is also known for his attempts at determining the distances to the Sun and the Moon (the first, unlike the second, not very successfully) and to observe that the lengths of the seasons are not equal, i.e. that the equinoxes and the midsummer and midwinter was not uniformly placed. This is now explained by the ellipticity of the Earth's orbit, but a simpler model would have been to place the Sun not exactly at the center of the orbital circle. He is also known for having compiled a catalogue of the positions of the fixed stars, which necessarily as noted must be continually updated.

[88]The fixed stars are not fixed but possess so called proper motions, but even the most mobile of the all, the arrow star of Barnard, a nearby red dwarf, does not move by more than $10''$ a year, hence much less than what is caused by the precession.

[89]As noted it has been a lucky coincidence that there has been for much of Human civilization a relatively bright star close to the pole.

[90]Say an orange sliced into two.

[91]Could there have been camera obscuras during the pre-historic age which left records such as cave paintings? One may easily imagine a rockwall, only exposed to sunlight through a narrow crack, which could be further reduced by human intervention. Such a setup would for a brief time cast an image on the wall. But to what purpose would such an image have served? The fascination of cave art is due to the maturity with which animals have been depicted with a feeling for their form and movements which has nothing to do with accurate reproduction of a mechanical nature. The point of a Camera Obscure only becomes apparent when we try to depict manmade objects defined by straight lines. Now what advantage is there to get an image on a flat surface? Is it not as difficult to copy it as it is to copy directly from nature? Or is the fact that it its perception is conceived on a flat surface of a psychological significance? One may of course copy directly on the surface on which the image is cast. The problem of making the image permanent by letting the light hitting the surface and the copying identical is the true problem of photography (as indicated by the name). It is a problem not of optics but of chemistry.

[92]In the real case if the point lies inside the conic there will be no real tangents at all, nor need a line intersect a conic in two points. But in the first case the tangents are complex conjugate as are the tangency points, hence the line that joins them is invariant under complex conjugation and hence real. Similarly the intersection points are complex conjugate and thus the corresponding tangents, which then necessarily intersect in a point invariant under complex conjugation and hence real. In the case of the unit circle this corresponds to nice formulas. To any point with distance $r > 0$ to the center we correspond the line perpendicular to the radius on which the point lies at distance $\frac{1}{r}$ from the center. Conversely given a line at distance r from the center we correspond the point with distance $\frac{1}{r}$ lying on the line through the center perpendicular to the given. The polar of the center will be the line at infinity, and conversely. And the polar of a line through the center will be a point at infinity, corresponding to the orthogonal direction of the line. There is also on each line a natural point, namely the one closest to the center, in this way we get a correspondence between points outside the center, known as reflection in a cirle or inversion with respect to it.

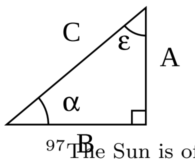
[93]but actually in 4-space.

[94]We have the notion of blowing up a point. This means replacing a point with all the lines through it, meaning identifying antipodal directions. If we instead replace by directions, we get what is called a real oriented blow up, and this amounts to punching out a disc around the point, no longer having a manifold, but a circle as a boundary. If we identify opposite points we get the (real unoriented) blow up, which makes it into a manifold. Topologically it means glueing to the boundary circle a Moebius strip. Topologically it is refered to attaching a cross-cap. The real projective plane is obtained by making a (real) blow up of a sphere. This can be illustrated by stereographic projection when a sphere is place on a plane and we project from a point of the sphere onto the plane. If we want to look at the image of the

projection pointthis will depend on the direction, i.e. we have to blow it up. Then the points will go to the line at infinity, whose neighbourhood makes up a Moebius strip. We can thus blow-down the line at infinity, and obtain a sphere. A sphere is a compactification of the plane at just one point. This is analogous to what will happen if we want to extend inversion in a cirle to the center we need to blow it up, thus corresponding a point to each point at infinity. We now have an inversion which is defined on a (real) blow-up of the projective plane which is a Klein-bottle. Inversion will then be an involution on the Klein-Bottle which fixes a circle. A Klein-bottle will hence be formed by gluing two Moebius strips along a common boundary. Had we used the oriented approach instead, the projective plane would have been replaced by a disc bounded by a circle, and the central point of the inversion, by another circle making up a cylinder, which one can see as part of the Klein-bottle (alternatively, the Klein-bottle can be seen as gluing Moebous strips to the ends of of a cylinder.

[95]Any function can in a unique way be expressed as the sum of an even and an odd function, the hyperbolic functions are the results of applying that to the exponential function.

[96]This actually gives a way to define the unit of length by measuring the angle a line at that distance extends.



In fact by using hyperbolic trigonometry we have

$$\cosh(C) = \cosh(A)\cosh(B)$$

(Pythagoras) and $\frac{\sin(\alpha)}{\sinh(A)} = \frac{1}{\sinh(C)}$ and the identity

$$\cosh^2(x) = 1 + \sinh^2(x)$$

we obtain $\sin^2(\alpha) = \frac{\sinh^2(C)-\sinh^2(B)}{\cosh^2(B)\sinh^2(C)}$. Now letting $\epsilon \to 0$ we have $\sinh(C) \to \infty$ and in the limit we get $\sin(\alpha) = \frac{1}{\cosh(B)}$.

[97]The Sun is of course not infinitely far away, but for all practical purposes when it comes to terrestrial optics it is.

[98]r will be a decreasing function of r if we center at the north pole, hence $r' < 0$.

[99]Is it true that if $\cup_n X_n = A$ that the 'measure' $\mu(A)$ is given by $\lim_{n\to\infty} \mu(X_n) = \mu(A)$?, this is a more subtle question for which the ancients were not ready yet to ponder. In fact it was not considered until the birth of modern measure theory at the end of the 19th century.

[100]Actually Grégoire de Saint-Vincent, a Flemish Jesuit. He was also the first to explicitly resolve Zeno's paradox by poiunting out that the time intervals formed a geometric series and added up to a finite sum, hence that Achilles would not catch up with the Turtle, before that time. His work refered to above was done in co-operation with his student de Sarasa (1618-67), who clarified the result and made explicit the connection with logarithms

[101]As we now the series only converges for $-1 < x \leq 1$ and only conditionally at the right endpoint. Unles x is rather small, the convergence is rather slow, on the other hand if $a > 1$ is small, we can compute $\log(a^n) = n\log(a)$ easily if we want to make tables. As an illustration let us look at $x = 0.1$, we get an accuracy of five decimal places if we include terms up to fourth order. Thus $\log(1.1) = 0.09531\ldots$ and hence $\log(1.4641) = 0.38134\ldots$, now $\sqrt{2} = 1.4142\ldots$ as $1.4641/1.4142 = 1.035285$ whose log is easily computed to 0.034677, subtracting we get 0.34666 and thus $\log(2) = 0.69333\ldots$

[102]There is of course nothing paradoxical, an infinite area can be covered with a finite amount of paint provided we allow the paint to be spread arbitrarily thin, and this is of coure what happens if we fill the infinite Toriicelli trumpet with paint, the thickness goes to zero as we we go to infinity.

[103]Hardly surprising he belonged to the circle around Mersenne. Stemming from a simple family of peasants he taught himself mathematics and changed his name to that of his birth-place (nowadays Senlis) to appear of aristicratic origins. He was reported to be hot-headed

and carried on disputes with Descartes filled with personal invectives. He is remembered by posterity for his invention of a special kind of scales now known as the Roberval balance.

[104]Let us find the normal to $y = \sqrt{x}$ at the point $(1,1)$. We want $(x-u)^2 + x = (1-u)^2 + 1$ should have a double root $x = 1$, hence $x^2 + (1-2u)x + 2u - 2 = x^2 - 2x + 1$ with the solution $u = \frac{3}{2}$.

[105]A Dutch mathematician who wrote a short treatise which would appear in a Latin edition of Descartes' *Geometry*.

[106]He was not only active as a mathematician but also as an astronomer having invented a special reflecting telesscope. He was the youngest child of an Episcopalian minister in Scotland. His interest in mathematics was transmitted by his mother who had had an uncle who had studied with Viète. He went to London in 1663 then the following year he made a continental tour for four years staying mostly in Italy, and returned via France and Flanders. Upon his return he was elected fellow of the Royal Society and became professor at St Andrews and Edinburgh respectively. At the latter he only stayed for a year, suffering a stroke while observing the Moons of Jupiter dying a few days later. He was the brother of the much more longlived David Gregory (1627-20) who was a physician and inventor. James Gregory is also known for having found infinite series expansions of the trigonometric functions.

[107]Was a British theologican and mathematician and the teacher of Newton. As is well-known he recognized the stature of this prime pupil and stepped down from his chair as a consequence freeing it for a worthier occupant. In the late fifties he took his continental tour taking him all the way to Constantiople. He was noted for his phyical courage, his lean constitution and pale complexion and was an inveterate smoker and slovenly dresser. He never married and ended up buried in Westminster Abbey. He was an eloquent preacher as can be surmised from his surviving sermons. His work was a Latin edition of Euclid, but also studed in detail Apollonius and Archimedes. He was also the first to write down an explicit primitive (i.e. in closed form) for the secant function, crucial for the Mercator projection. Much of his mathematical work was taken up by optics.