## Some combinatorial questions

- How many permutations $a_1 a_2 \cdots a_n$ of $[n] := \{1, 2, ..., n\}$ have exactly $k$ *descents*, that is, how many of these permutations have $a_i > a_{i+1}$ for exactly $k$ values of $i$?
- Triangulate the unit cube in $n$ dimensions by partitioning it into simplices of the form

  $$\{(x_1, x_2, \ldots, x_n) \mid 0 \leq x_{a_1} \leq x_{a_2} \leq \cdots \leq x_{a_n} \leq 1\},$$

  where $a_1 a_2 \cdots a_n$ is a permutation of $[n]$. What can be said about the number of maximal intersections between two such simplices?
- If $P_n$ is the Boolean algebra on $n$ atoms, seen as a partially ordered set, in how many ways can one pick two distinct maximal chains in $P_n$ that intersect maximally, that is, in all but one element? (A chain is a set of pairwise comparable elements.)

The answer (or, rather, one answer) to the first question is the coefficient to $t^{k+1}$ in the *Eulerian polynomial* $A_n(t)$ that can be defined by the following identity:

$$\sum_{k=0}^{\infty} k^n t^k = \frac{A_n(t)}{(1-t)^{n+1}}.$$

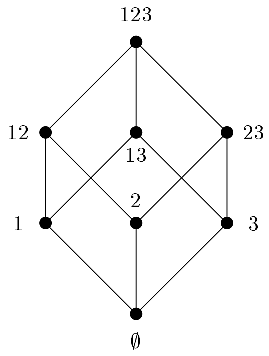
If we modify the latter two questions slightly it turns out that they have the same answer.

Here are the polynomials $A_n(t)$ for $n = 1, 2, \ldots, 5$. To the right we have written the coefficients of each polynomial:

$$\begin{array}{c}
t \\
t + t^2 \\
t + 4t^2 + t^3 \\
t + 11t^2 + 11t^3 + t^4 \\
t + 26t^2 + 66t^3 + 26t^4 + t^5
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 4 & & 1 & & \\
 & 1 & & 11 & & 11 & & 1 & \\
1 & & 26 & & 66 & & 26 & & 1
\end{array}$$

Here is a list of the permutations of three elements, and the Hasse diagram of the Boolean algebra of three atoms:

1 2 3
1 3 2
2 1 3
2 3 1
3 1 2
3 2 1




It is easy to see how a maximal chain corresponds to a permutation. For example, the maximal chain $\emptyset < \{3\} < \{1, 3\} < \{1, 2, 3\}$ corresponds to the permutation 312.

Order now the maximal chains by the above (lexicographic) order of their corresponding permutations. Given a maximal chain $C$ with corresponding permutation $\pi$, can you see why the number of maximal chains preceding $C$ and intersecting $C$ maximally equals the number of descents in $\pi$?

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Another interesting aspect of the Eulerian polynomials $A_n(t)$ is their *exponential generating function.* Namely, making $A_n(t)$ the coefficient of $x^n$ in an exponential power series gives the following pretty identity:

$$\sum_{n\geq 0} A_n(t)\frac{x^n}{n!} = \frac{(1-t)e^{x(1-t)}}{1-te^{x(1-t)}}.$$