New problems added at the end!

1. (a) Find the number of permutations $a_1 a_2 \dots a_n$ in $\mathcal{S}_n$ such that $|a_i - i| \leq 1$ for each i.

   (b) According to one or more other exercises here, the answer in (a) can be written as a certain nice sum of approximately $n/2$ binomial coefficients. Is there a refinement of the problem in (a) that reflects this?

2. Place n points on a circle and draw the chords between each pair of points. Assume that the points are in general position, that is, no three chords intersect in a point. Determine the number of regions inside the circle (Give a simple formula and prove it combinatorially).

3. Give combinatorial proofs of the following identities:

   (a) $\sum_i \binom{k}{i}\binom{n-k}{d-i} = \binom{n}{d}$.

   (b) $\sum_k (-1)^{n-k}\binom{n}{k}\binom{k}{m} = \binom{0}{n-m}$.

   (c) $\sum_k \binom{n}{k}^2 = \binom{2n}{n}$. Look at your proof here. Do you see any generalizations that require only trivial modifications of the proof?

4. Let $c(n,k)$ be the number of compositions of n whose greatest part is at most k. Show that $\sum_{n\geq 0} c(n,k)x^n = \frac{1-x}{1-2x+x^{k+1}}$. You should set $c(0,k) = 1$ for all k

5. Prove the following identity via generating functions. Can you find a combinatorial proof (of a suitably modified version of this identity)?

$$\sum_{k\geq 0} \binom{2n-k}{n}\left(\frac{1}{2}\right)^{2n-k} = 1.$$

6. (a) Find the *bivariate* generating function for the binomial coefficients, that is,

$$\sum_{n,k} \binom{n}{k} x^n y^k$$

   (it should be of the form $1/(1-P(x,y))$).

   (b) Set $y = x$ in (a) to get a well-known generating function.

7. (a) A *multi-set* is a set with repeated elements allowed. How many n-element multi-sets are there with elements from $[2k]$ such that the numbers $1, 2, \dots, k$ appear at most once each and the numbers $k+1, k+2, \dots, 2k$ appear an even number of times each? For $n = 3$ and $k = 2$: 133, 144, 233, 244

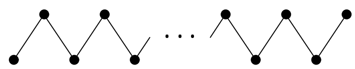
   (b) Show, combinatorially, that the answer equals the number of monomials of degree n in k variables.

8. How many permutations $a_1 a_2 \cdots a_n$ satisfy the following?: If $2 \leq j \leq n$ then $|a_i - a_j| = 1$ for some i with $1 \leq i < j$. Give a combinatorial proof. 123, 213, 231, 321

9. (a) Let $f(n)$ be the number of ways of placing n identical balls in n numbered boxes so that no box gets more than two balls. Find the generating function $\sum_n f(n)x^n$.

   (b) What do the coefficients of $(1+x+x^2)^n$ count?

   (c) Write out a triangle with the coefficients of $(1+x+x^2)^n$ for $n = 1, 2, \ldots$. The diagonals seem to grow polynomially. Do they? What are the polynomials?

10. Let P_n be the n-element *fence poset*: 

   (a) In how many different ways can P_n be partitioned into k disjoint chains?

   (b) Determine the total number of ways of partitioning P_n into disjoint chains.

11. Let $B_1, B_2, \ldots, B_m$ be the blocks of a partitioning of $[n]$. We say that B_i and B_j *overlap* if $\min(B_i) < \max(B_j)$ and $\min(B_j) < \max(B_i)$. Find the number of partitionings of $[n]$ have exactly one pair of overlapping blocks. The formula should be simple and (of course!) proved combinatorially.

0, 0, 1, 6, 24, 80, 240, 672, 1792...

12. A *Dyck path* from $(0,0)$ to $(2n, 0)$ is a lattice path with steps $(1,1)$ and $(1,-1)$ that never goes below the x-axis. Show combinatorially that the number of such paths is the n-th *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$.

You may rewrite C_n if necessary

13. A random walk in one dimension is a (countably) infinite sequence of 1's and -1's (each having probability $1/2$), indicating unit steps on the real line in the positive and negative directions, respectively.

    The probability that a random walk W, starting at 0, returns to 0 clearly equals $\sum_{n\geq 1} p_n$ where p_n is the probability that W returns to 0 *for the first time* after exactly $2n$ steps. By counting the number of walks that return after exactly $2n$ steps, and using the appropriate generating function, show that a random walk returns to 0 with probability 1.

14. Let $R(x) = \frac{P(x)}{Q(x)}$ be a rational function (P and Q polynomials) such that $Q(0) \neq 0$. Show that $R(x) = S(x) + T(x)$, where T is a polynomial and

$$S(x) = \frac{a_1}{(1-b_1x)^{e_1}} + \frac{a_2}{(1-b_2x)^{e_2}} + \cdots + \frac{a_n}{(1-b_nx)^{e_n}},$$

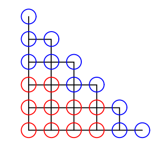
    and where a_i, b_i and e_i are constants. (The point is that the generating function $\frac{a}{(1-bx)^e}$ has nice coefficients). Hint: Let $Q(x) = q_0 + q_1x + \cdots + q_nx^n$ and $Q_R(x) = q_n + q_{n-1}x + \cdots + q_0x^n$. What is the relation between the roots of Q and Q_R? Use this to write $Q(x) = q(1-r_1x)(1-r_2x)\cdots(1-r_nx)$.

15. There are $d!$ paths of length d along edges from the origin to the vertex $(1, 1, \ldots, 1)$ in the unit $d-$cube. How many non-self-intersecting paths are there of length k? Non-intersecting means that the path doesn't visit the same vertex twice. (Hard. Probably unsolved. Maybe hopeless.)

A path is self-intersecting if it visits the same vertex twice

16. Determine the number of $2 \times n$ matrices of 0's and 1's with no adjacent 0's (in rows or columns). $1 \times n$: Fibonacci

17. Consider the set T of integer points (a, b) in the plane with non-negative coordinates such that $a + b < n$.

    Let S be an arbitrary subset of T. Color the points in S red, and also all points (x, y) such that $x \leq a$ and $y \leq b$ for some point (a, b) in S. Color the remaining points of T blue.

    A subset of T is said to be *horizontal* if it has n blue elements, no two of which have the same x-coordinate. A subset of T is said to be *vertical* if it has n blue elements, no two of which have the same y-coordinate.

    Prove, preferably combinatorially, that there are as many vertical as horizontal subsets in T.

    Future problem: The red set above is an *order ideal* in the poset $\mathbb{N}^2$ with the ”usual” order. What sort of generalization can you think of?

18. Let K_n be the set of weakly increasing sequences, of length n, consisting of natural numbers not exceeding n, whose sum is a multiple of $(n + 1)$. Prove that K_n contains C_n elements, where C_n is the n-th Catalan number. For $n = 3$ the sequences are:

    $$000, \quad 013, \quad 022, \quad 112, \quad 233.$$

    Can you give a nice bijection to Dyck paths?

    (I don't think this problem is on Stanley's list (see link to Enumerative Combinatorics), but if you find the solution there, that's cheating!).