19. Let D be the differential operator d/dx. State and prove a theorem about the coefficients (to x^kD^i)in the expansion of $(xD)^n$. Can you give some sort of explanation, that is, a combinatorial proof?

$(xD)^2 = x(D(xD)) = x(D + xD^2)$

20. Prove the following identity for the Stirling numbers of the second kind by an inclusion/exclusion argument:

Inclusion/Exclusion: See p. 110 in gfology by Wilf, or link on homepage

$$S(n,k) = \sum_{i=1}^{k} (-1)^{k-i} \frac{i^{n-1}}{(i-1)!(k-i)!}.$$

Maybe you should first modify the identity slightly.

21. Let $A(n,k)$ be the number of permutations in the symmetric group $\mathcal{S}_n$ with exactly k *excedances*. An excedance in $\pi = a_1a_2\cdots a_n$ is an i such that $a_i > i$. Find a recurrence for $A(n,k)$.

22. Given a permutation $\pi \in \mathcal{S}_n$, we can code its excedances with an *ab-word*, that is, a word $w = x_1x_2\cdots x_n$ where $x_i = b$ if i is an excedance in π and $x_i = a$ otherwise. Let $[w]$ be the number of permutations in $\mathcal{S}_n$ whose excedance word is w. Show that if v and w are any ab-words, then $[wbav] = [wabv] + [wav] + [wbv]$.

    Conclude that $[w]$ is odd for all w.

23. A *Left-to-right minimum* in a permutation $\pi = a_1a_2\cdots a_n$ is an i such that $a_i < a_j$ for all $j > i$. Let $N(S,n)$ be the number of permutations in $\mathcal{S}_n$ whose set of Left-to-right minima is S. Example: $N(\{2,4\},4) = 3$, viz. 4132, 3142, 2143.

    Note that the LtoR-min is identified with the *place*, not the letter

    Use this to give a combinatorial proof of the fact that

$$\sum_{S\subseteq[n-1]} \Pi(S) = n!,$$

where $\Pi(S) = \prod_{k\in S} k$. Example:

$$1 + (1+2+3) + (1\cdot 2 + 1\cdot 3 + 2\cdot 3) + (1\cdot 2\cdot 3) = 4!.$$

24. Let $S(n,k)$ and $A(n,k)$ be the Stirling and Eulerian numbers, respectively. Prove combinatorially that

$$S(n,k) = \frac{1}{k!}\sum_i \binom{i}{n-k} A(n,i).$$

25. Invert the identity in the previous exercise (and the combinatorial proof).

26. The signless Stirling numbers of the first kind have the generating function

$$\sum_k c(n,k)x^k \quad = \quad x(x+1)(x+2)\cdots(x+n-1).$$

Find a generating function F_n that specializes to the above with suitable substitutions and such that F_n actually *generates* all permutations of $[n]$ with k cycles. That is, each term in F_n should correspond (in some simple way that you explain) to a unique permutation with k cycles.

27. Given a deck of cards, play the following game: At each stage, turn up a new card and place it on any higher card on the table, if there is one. Otherwise, start a new pile.

Now play this game with the letters of a permutation of $[n]$, in such a way that we always place a letter in the leftmost pile possible.

(a) Describe the number of resulting piles in terms of properties of the permutation.

(b) How many cards will there be in the first (leftmost) pile? Let $R(\pi)$ be this number. What is the distribution of $R(\pi)$ over $\mathcal{S}_n$, that is, for how many permutations $\pi \in \mathcal{S}_n$ is $R(\pi) = k$?
Hint: These are well-known numbers.

28. Let $f(n)$ be the number of fixed-point-free involutions in $\mathcal{S}_{2n}$. Find $\sum_n f(n)x^n/n!$.

29. Let c_n^k be the number of partitions of $[2n]$ into k blocks of even sizes. Then

$$E_{2n} = \sum_{k=1}^{n} (-1)^{n-k} \cdot k! \cdot c_n^k,$$

where the E_{2n} are the *Euler numbers* (not to be confused with the Eulerian numbers), which count the alternating permutations in $\mathcal{S}_{2n}$, that is, permutations $a_1a_2\cdots a_n$ such that $a_1 > a_2 < a_3 > \cdots$. Give a combinatorial proof. The proof (slightly modified) should also cover the following identity:

$$E_{2n-1} = \sum_{k=1}^{n} (-1)^{n-k} \cdot (k-1)! \cdot c_n^k.$$

30. (a) Let $\pi = a_1a_2\cdots a_{n-1} \in \mathcal{S}_{n-1}$. Analyze the effect on $\text{MAJ}\,\pi$ of inserting n in the n different places in π. The result is called the MAJ-*coding* of π. Do you see a pattern? Prove it. For definitions, see below

(b) Show that $\sum_{\pi\in\mathcal{S}_n} q^{\text{MAJ}\,\pi} = \sum_{\pi\in\mathcal{S}_n} q^{\text{INV}\,\pi}$

(c) How can the result in (a) be used to construct a bijection $\phi : \mathcal{S}_n \to \mathcal{S}_n$ such that $\text{INV}\,\pi = \text{MAJ}\,\phi(\pi)$?

31. Let $A_n(t)$ be the n-th Eulerian polynomial. It is well known that when n is odd, $\pm A_n(-t)$ equals the number of *alternating permutations* in $\mathcal{S}_n$, that is, permutations $a_1a_2\cdots a_n$ with $a_1 > a_2 < a_3 > \cdots$. Give a combinatorial proof. Hint: The trick here is to find the right representation of the permutations.

32. The coefficients of the Eulerian polynomials are *log-concave*, that is, we have $A(n,k)^2 \geq A(n,k-1)A(n,k+1)$ for all k. This implies that they are unimodal, that is, they increase to a maximum and then decrease (we can't have $A(n,k-1) > A(n,k) < A(n,k+1)$ for any k). Find a (new!) combinatorial proof. (Very hard).

A *descent* in a permutation $\pi = a_1a_2\cdots a_d$ is an i such that $a_i > a_{i+1}$.

The *major index* of π, $\text{MAJ}\,\pi$, is the sum of the descents in π.

An *inversion* in π is a pair (i,j) such that $i < j$ and $a_i > a_j$.

EXAMPLE: If $\pi = 7153642$ then the descents in π are 1, 3, 5 and 6, so $\text{MAJ}\,\pi = 1+3+5+6 = 15$. There are $6+0+3+0+2+1+0 = 12$ inversions in π.