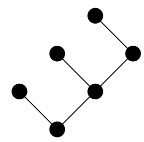
33. Let P be an n-element poset. For $x \in P$, let $I_x = \#\{y \in P \mid y \leq x\}$. Let $e(P)$ be the number of linear extensions of P. Show that linear extension: order preserving map from P to a chain of size $|P|$

$$e(P) \geq \frac{n!}{\prod_{x \in P} I_x}.$$

34. Let P_n be the poset with elements $x_1, \ldots, x_n, y_1, \ldots, y_n$, defined by $x_i < x_{i+1}$ and $x_i < y_i$ for all i. For example, P_3 is shown in the margin. 10 point problem



a) Give a simple formula for the rank-generating function $R(J(P_n), q)$.
   $J(P)$ is the *lattice of order ideals* in P (ordered by inclusion).
   $R(G, q)$ is the number of elements of rank q in a graded poset G.
b) Let $P = \lim_{n\to\infty} P_n$. Determine $R(J(P), q)$.
c) Give a simple formula for $e(P_n)$, the number of linear extensions of P_n.
d) Let $\Omega(P_n, k)$ be the order polynomial for P_n. Express $\Omega(P_n, k)$, for $k \in \mathbb{P}$, in terms of the Stirling numbers of the second kind.
e) Express $\Omega(P_n, -k)$ in terms of the Stirling numbers of the first kind.

35. How many maximal chains are there in Π_n, the lattice of all partitions of $[n]$?

36. Let P be a finite poset and let μ be the Möbius function of $\widehat{P} = P \cup \{\hat{0}, \hat{1}\}$. Suppose P has a fixed- point-free automorphism $\phi : P \to P$ of prime order p (that is, $\phi(x) \neq x$ and $\phi^p(x) = x$ for all $x \in P$). Show that $\mu(\hat{0}, \hat{1}) \equiv -1 \pmod{p}$. What is this result called in the case when $P = \Pi_p$, the partition lattice? automorphism: order-preserving bijection

37. Let E_n be the poset (ordered by inclusion) of all subsets of $[n]$ whose elements have even sum. $\emptyset$, 2, 13, 123

a) Compute $\#E_n$.
b) Compute $\mu(S, T)$ for all S and T in E_n. Hint: "Rank-Selection". But think first.
c) (Hopeless?) Do b) for all subsets whose elements have sum divisible by k.

38. Do there exist any infinite antichains in the lattice $\mathbb{N}^2$ ordered by Zorn's Lemma?

$$(x, y) \leq (z, w) \iff x \leq z \text{ and } y \leq w?$$

39. (a) Let P be a finite poset with a least element. If $f : P \longrightarrow P$ is an order preserving map, show that f has a fixed point.
    (b) Define an element $x \in P$ to be *central* if x is comparable to every element in P. Generalize (a) to show that if P has a central element then every o.p. self-map of P has a fixed point.

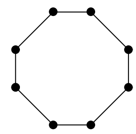
(c) Give an example of a poset P with no central element such that every o.p. self-map of P has a fixed point.

Note: A thereom by Baclawski and Björner (*Adv. in Math.* **31** (1979)), says that every self-map of P has a fixed point if the order complex of P is *acyclic*, i.e. if the (reduced) homology groups of the complex are trivial. If P has a central element c then the order complex $\Delta(P)$ of P is a *cone complex* with *apex* c—i.e. every maximal simplex contains c—so the topological space realizing $\Delta(P)$ is contractible (it can be contracted onto c) and hence $\Delta(P)$ is acyclic.

40. A theorem of Stanley's (Discrete Math, 1973) gives a combinatorial interpretation of the values of the chromatic polynomial of a graph at negative integers. A particular case is $\chi_G(-1)$ which, up to a sign, equals the number of acyclic orientations of G. Can you find a simple combinatorial proof of this?

    Hint(?): How many acyclic orientations are there of the complete graph on n vertices?

41. Compute the chromatic polynomial $\chi_k(n)$ of the k-cycle C_k. Can you find a direct proof (assuming you give a simple answer)?

    8-cycle

    Define the W-polynomial of C_k by

$$\sum_{n\geq 0} \chi_k(n)x^n = \frac{W_k(x)}{(1-x)^{k+1}}.$$

    Can you find a simple formula for the coefficients of the W-polynomial? In particular, can you show that it is symmetric if k is odd?

42. The numbers in the following triangle are defined recursively. The numbers on the edges are the Euler numbers. The numbers in between also count alternating permutations, but on a finer scale, that is, subject to some restrictions. Interpret these numbers combinatorially and show that they satisfy the recurrence defining the triangle. Hint: Look at the last letter of each permutation.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 0 & & & & \\
 & & & 0 & & 1 & & 1 & & & \\
 & & 2 & & 2 & & 1 & & 0 & & \\
 & 0 & & 2 & & 4 & & 5 & & 5 & \\
16 & & 16 & & 14 & & 10 & & 5 & & 0 \\
 & & & & & \vdots & & & & &
\end{array}$$

    This triangle appears in V.I. Arnold: *Bernoulli-Euler updown numbers associated with function singularities, their combinatorics and arithmetics*, Duke Math. J. **63** No. 2 (1991), 537-555. Arnold states that each line in the triangle defines *finite mass distributions* and he shows, among other things, that the Euler number E_d is the number of *maximal morsifications* of the function x^{d+1}.