A combinatorial proof (by example) of $\sum_k S(n,k)s(k,m) = \delta_{nm}$.

The identity is equivalent to $\sum_k (-1)^{n-k} S(n,k)c(k,m) = \delta_{nm}$, where $c(k,m)$ is the signless Stirling number of the first kind.

Fix n and m.

1. Write every partition of $[n]$ so that the elements of each block are in increasing order, and order the blocks in order of decreasing minima.

   Example: 9–47–236–158.

2. Write a permutation of $[m]$ in standard form, that is, least element of each cycle first and the cycles in order of decreasing minima.

   Example: (578)(364)(2)(19)

3. For each k-partition p of $[n]$ we will treat the blocks of p as the numbers in $[k]$ and construct a permutation of the blocks. We therefore need a total order on the blocks. Order the blocks by least element.

   Example: $P = 9{-}47{-}236{-}158$ and $\pi = (b)(acd)$. This gives $(236)(158, 47, 9)$ and we can write it as $236158479 = W(P, \pi)$, but then we don't know where it came from. The word W could just as well have come from $(2, 36)(158, 4, 79)$. We can not recover the blocks within each cycle, but we can recover the cycles, because a letter in W starts a new cycle if and only if it is smaller than all the preceding letters in W.

4. This describes a unique (but not bijective) way of constructing a permutation (word) from each pair (P, π) where P is a k-partition of $[n]$ and $\pi \in \mathcal{S}_k$.

5. If we can show that, whenever $n \neq m$, each such word arises as many times with a positive sign as it does with a negative sign when we do this for each $k \in [m, n]$ then we're done.

6. How can, for example, $W = 236158479$ arise? There are only two cycles, and the second cycle must start with 1. The 8 and the 4 cannot have belonged to the same block in p, but apart from that we can split each of the cycles into blocks at any place except for the 84, since that is the only place where a number is followed by a smaller one within a cycle

7. Thus, W could have arisen from any partition obtained by splitting $(2 - 3 - 6)(1 - 5 - 84 - 7 - 9)$ at any subset of the dashes. The number of chosen dashes determines the parity of $(n - k)$ (the parity is either always the same or always different.)

8. The number of times W arises is 2^d where d is the number of dashes and the total contribution of W in the sum is

   $$\sum_i (-1)^i \binom{d}{i} = \delta_{d,0}.$$

   This sum is zero, unless $d = 0$. But in that case, the numbers in each cycle must be decreasing, so each cycle contains only one number, which means that $n = m$.