The Catalan numbers

Definition 1 *The n-th* Catalan number, *C_n, is the number of paths from $(0,0)$ to $(2n,0)$, with steps of type $(1,1)$ and $(1,-1)$ that never go below the x-axis.*

Theorem 2 *For all $n \geq 1$ we have $C_n = \sum_k C_{k-1}C_{n-k}$.*

Corollary 3 *The Catalan numbers have the generating function*

$$\sum_{n\geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}.$$

Proof:

$$\begin{aligned}
\sum_{n\geq 0} C_n x^n &= 1 + \sum_{n\geq 1} C_n x^n = 1 + \sum_{n\geq 1} x^n \sum_k C_{k-1}C_{n-k} \\
&= 1 + x\sum_{n\geq 1} x^{n-1} \sum_k C_{k-1}C_{(n-1)-(k-1)} \\
&= 1 + x\sum_{n\geq 0} x^n \sum_k C_{k-1}C_{n-(k-1)} \\
&= 1 + x\sum_{n\geq 0} x^n \sum_k C_k C_{n-k} = 1 + x\left(\sum_{n\geq 0} C_n x^n\right)^2,
\end{aligned}$$

so, if we let $F(x) = \sum_{n\geq 0} C_n x^n$, we have $xF(x)^2 - F(x) + 1 = 0$, which gives the solution claimed. (That we must choose the minus sign in front of the root follows from $F(0) = 1$.) □

Definition 4 *If α is a real number and $n \geq 0$ an integer, then*

$$\binom{\alpha}{n} = \frac{\alpha\cdot(\alpha-1)\cdot\cdots\cdot(\alpha-n+1)}{n!}.$$

Note: $\binom{\alpha}{0} = 1$, for then the product in the numerator is empty

Theorem 5 (Binomial Theorem) *For all $\alpha \in \mathbb{R}$ we have*

$$(1+x)^\alpha = \sum_{n\geq 0}\binom{\alpha}{n}x^n.$$

Proof: The right hand side is the Taylor expansion of the left hand side about 0. □

Lemma 6 *We have* $\displaystyle\frac{1}{\sqrt{1-4x}} = \sum_{n\geq 0}\binom{2n}{n}x^n.$

Proof: We have

$$(1-4x)^{-1/2} = \sum_{n\geq 0} \binom{-1/2}{n}(-4x)^n = \sum_{n\geq 0} (-1)^n \binom{-1/2}{n} 4^n x^n.$$

It therefore suffices to show that $\binom{2n}{n} = (-1)^n\binom{-1/2}{n}4^n$, which is left to the interested reader. □

Theorem 7 *The Catalan numbers satisfy* $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$.

Proof: If we integrate both sides in Lemma 6 and divide by x we get

$$\sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^n = \frac{1-\sqrt{1-4x}}{2x} + C,$$

where C is a constant. This constant is 0, because $\frac{1}{0+1}\binom{2\cdot 0}{0} = 1$ and the fraction in the RHS has the limit 1 as $x \to 0$. This, together with Corollary 3, proves the claim. □