

Chromatic polynomials

**Definition 1** Let  $G$  be a finite graph with vertex set  $V$ . A *coloring of  $G$  with  $n$  colors* is a map  $f : V \rightarrow \{1, 2, \dots, n\}$  such that  $f(x) \neq f(y)$  if  $x$  and  $y$  are adjacent in  $G$ .

**Definition 2** Let  $G$  be a finite graph. The *chromatic polynomial of  $G$*  is the function  $\chi_G(n) =$  number of ways to color  $G$  with  $n$  colors.

Theorem:  
The chromatic  
polynomial is  
a polynomial  
in  $n$

Observe that the *chromatic number* of  $G$ ,  $\chi(G)$ , is equal to the least positive integer  $p$  for which  $\chi_G(p)$  is greater than 0.

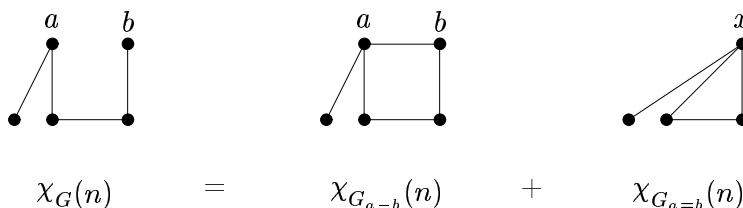
Computing the chromatic polynomial of a graph, or even the chromatic number, is an NP-complete problem.

Here is one way to determine the chromatic polynomial of a graph  $G$ :

**Theorem 3** Given a graph  $G$ , and two different vertices  $a, b$  in  $G$ , so that  $(a, b)$  is not an edge in  $G$ , let  $G_{a-b}$  be the graph obtained by adding an edge between  $a$  and  $b$  in  $G$ , and let  $G_{a=b}$  be the graph obtained by replacing  $a$  and  $b$  by a vertex  $x$  so that  $x$  has an edge to each vertex to which  $a$  or  $b$  has an edge in  $G$ . Then

$$\chi_G(n) = \chi_{G_{a-b}}(n) + \chi_{G_{a=b}}(n).$$

**Proof:** Each coloring of  $G$  where  $a$  and  $b$  have different colors corresponds to a unique coloring of  $G_{a-b}$  and each coloring of  $G$  where  $a$  and  $b$  have the same color corresponds to a unique coloring of  $G_{a=b}$ . □



Observe that the recursive formula in Theorem 3 can also be written thus:

$$\chi_{G_{a-b}}(n) = \chi_G(n) - \chi_{G_{a=b}}(n). \tag{1}$$

Using this version of the formula means removing edges from the graph until there are only isolated vertices left. Which of these two formulas is more effective in computing  $\chi_G$  depends on  $G$ 's structure, in particular on whether it has few or many edges.

**Definition 4** Let  $G$  be a graph and  $M$  a set of vertices in  $G$ . The set  $M$  is *stable* (or *independent*) if no two vertices in  $M$  are adjacent in  $G$ .

**Definition 5** The  $i$ -th *falling factorial* of  $n$  is  $(n)_i = n(n-1)(n-2) \cdots (n-i+1)$ . We let  $(n)_0 = 1$ .

A coloring of  $G$  with exactly  $k$  colors always induces a partition of the vertices in  $G$  into  $k$  stable sets, because each set of like-colored vertices must be stable. Conversely, every partition of  $G$  into  $k$  stable sets gives a coloring of  $G$  with  $k$  colors, by assigning one color to each stable set. If we have  $n$  colors to choose from then this can be done in  $n(n-1)(n-2) \cdots (n-k+1)$  different ways, that is, in  $(n)_k$  different ways. This leads to the following theorem.

**Theorem 6** Let  $S_G(k)$  be the number of ways to partition the vertices of  $G$  into exactly  $k$  stable sets. Then

$$\chi_G(n) = \sum_k S_G(n)_k.$$

It follows directly that  $\chi_G(n)$  is a polynomial in  $n$ , since  $(n)_k$  is obviously a polynomial in  $n$  for each  $k$ . This also shows that  $\chi_G$  is monic and of degree  $d =$  number of vertices in  $G$ .