Lemma 1 *Let $f : [d] \longrightarrow [n]$. There is a unique permutation $A(f) = \pi = a_1 a_2 \cdots a_d$ so that*

(i) $f(a_1) \geq f(a_2) \geq \cdots \geq f(a_d)$, and

(ii) $f(a_i) > f(a_{i+1})$ if $a_i > a_{i+1}$.

Proof: For each $f : [d] \longrightarrow [n]$ there is a unique partition $B_1 - B_2 - \cdots - B_k$ of $[d]$ such that f is constant on each block of B_i and $f(B_i) > f(B_2) > \cdots f(B_k)$. The permutation π is obtained by writing the elements of B_1 in increasing order, then those of B_2 in increasing order and so on. □

Lemma 2 *For each $\pi = a_1 a_2 \cdots a_d$ with k descents there are exactly $\left(\!\binom{n-k}{d}\!\right)$ functions $f : [d] \longrightarrow [n]$ such that $\pi = A(f)$, where $A(f)$ is the permutation defined in Lemma 1.*

Proof: Choose d numbers, with repetitions allowed, from $[n-k]$ and label them $n_1, n_2, \ldots, n_d$ so that $n_1 \geq n_2 \geq \cdots \geq n_d$. Define n'_i by $n'_i = n_i + \operatorname{des}(a_i a_{i+1} \ldots a_d)$. Thus, $n'_i \in [n]$, since $n'_i = n_i + \operatorname{des}(a_i a_{i+1} \ldots a_d) \leq n_i + k \leq n$. Define $f : [d] \longrightarrow [n]$ by $f(a_i) = n'_i$. Then we have $f(a_i) = n'_i \geq n'_{i+1} = f(a_{i+1})$ and, if $a_i > a_{i+1}$ then $n'_i > n'_{i+1}$, so $f(a_i) > f(a_{i+1})$. Consequently, $A(f) = \pi$. Moreover, two different choices of the numbers $n_1, n_2, \ldots, n_d$ give two different functions f. Thus, we have exhibited $\left(\!\binom{n-k}{d}\!\right)$ different functions f with $A(f) = \pi$.

Conversely, let $f : [d] \longrightarrow [n]$ and suppose that $A(f) = \pi$ with $\operatorname{des}(\pi) = k$. Then we have $f(a_i) \geq f(a_{i+1})$, with inequality if $a_i > a_{i+1}$. Let $n'_i = f(a_i)$ and let $n_i = n'_i - \operatorname{des}(a_i a_{i+1} \ldots a_d)$. Then $n'_1 = f(a_1) \geq 1 + \operatorname{des}(\pi)$, so $n_1 = n'_1 - \operatorname{des}(a_i a_{i+1} \ldots a_d) \geq 1$, and $n'_1 \leq n$, so $n_1 \leq n - \operatorname{des}(\pi) = n - k$. Thus, $n_1 \in [n-k]$.

Now, if 1 is not a descent, then the same reasoning can be applied to n_2, to show that $n_2 \in [n-k]$. Otherwise, if 1 is a descent, so $a_1 > a_2$, then $\operatorname{des}(a_2 a_3 \ldots a_d) = \operatorname{des}(\pi) - 1$ and thus we have $n'_2 \geq k - 1$. So, again, we have that $n_2 \geq 1$ and , since $n'_2 < n'_1$, that $n_2 \leq (n-1) - \operatorname{des}(a_2 a_3 \ldots a_d) = (n-1) - (k-1) = n - k$. That is, $n_2 \in [n-k]$.

Analogous reasoning applies to each of the n_i's, because whenever i is a descent in π we have that $n'_i > n'_{i+1}$ and $\operatorname{des}(a_i a_{i+1} \ldots a_d) = \operatorname{des}(a_{i+1} a_{i+2} \ldots a_d) + 1$.

Thus we have established a one-to-one correspondence between the set

$$\{f \mid f : [d] \longrightarrow [n] \text{ and } A(f) = \pi\}$$

and the set of d-element multisets with elements from $[n-k]$, where $k = \operatorname{des}(\pi)$. □

The preceding lemma is essentially equivalent to the following theorem.

Theorem 3 *The Eulerian polynomials $A_d(t)$ satisfy the following identity:*

$$\sum_n n^d t^n = \frac{A_d(t)}{(1-t)^{d+1}}.$$

□