The Stirling numbers of the second kind have the generating function

$$\sum_n S(n,k)x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

$$= x^k(1+x+x^2+\cdots)(1+2x+4x^2+\cdots)\cdots(1+kx+(kx)^2+\cdots)$$

Now refine the above GF as follows:

$$\frac{x^k}{(1-x)(1-2x)\cdots(1-kx)} \to \frac{x_0}{1-x_1}\,\frac{x_0}{1-(x_1+x_2)}\cdots\frac{x_0}{1-(x_1+x_2+\cdots+x_k)} \quad (*)$$

$$= (x_0+x_0x_1+x_0x_1^2+x_0x_1^3+\cdots)(x_0+x_0(x_1+x_2)+x_0(x_1+x_2)^2+\cdots)\cdots$$

$$(x_0+x_0(x_1+x_2+\cdots+x_k)+x_0(x_1+x_2+\cdots+x_k)^2+\cdots)$$

where the x_i's do *not* commute. A typical monomial, say $x_0x_1x_1x_0x_0x_1x_3x_3x_2$ (where $k=3$), is translated into a partition as follows. Write the numbers from 1 to n in order under the x_i's:

x_0	x_1	x_1	x_0	x_0	x_1	x_3	x_3	x_2
1	2	3	4	5	6	7	8	9

Each x_0 means that the number below it starts a new block. Otherwise, x_i means that the number below it goes into the ith block. Here, 1 starts a new block (as always) and then 2 and 3 go into that same block. New blocks are started by both 4 and 5 and then 6 goes into the first block, whereas 7 and 8 go into the third one and 9 into the second one. This yields the partition $1236-49-578$.

Setting $x_i = x$ for all i gives the original generating function.

Observe that we could also replace the x_0's in $(*)$ by x_i's to get

$$\frac{x^k}{(1-x)(1-2x)\cdots(1-kx)} \to \frac{x_1}{1-x_1}\,\frac{x_2}{1-(x_1+x_2)}\cdots\frac{x_k}{1-(x_1+x_2+\cdots+x_k)}$$

and then simply assign each of the letters in $[n]$ to a block as above. Then the first occurrence of x_i shows which is the least element in block i.

In fact, if we then erase the x's, leaving a sequence of numbers, we get all sequences of length n of integers from $[k]$ satisfying the following condition:

> For all $i > 1$, the first occurrence of i is preceded by some occurrence of $i-1$.

Such sequences have been called *restricted growth functions* and are clearly in one-to-one correspondence with partitions of $[n]$ into k blocks.