COUNTEREXAMPLES TO THE NEGGERS-STANLEY CONJECTURE

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Abstract. The Neggers-Stanley conjecture (also known as the Poset conjecture) asserts that the polynomial counting the linear extensions of a partially ordered set on $\{1, 2, \ldots, p\}$ by their number of descents has real zeros only. We provide counterexamples to this conjecture.

1. The Neggers-Stanley Conjecture

Let P be a poset (partially ordered set) on $[p] := \{1, 2, \ldots, p\}$. We will use the symbol $<$ to denote the usual order on the integers and the symbol $\prec$ to denote the partial order on P. The *Jordan-Hölder set*, $\mathcal{L}(P)$, of P is the set of permutations π of $[p]$ which are linear extensions of P, i.e., if $i \prec j$ then $\pi^{-1}(i) < \pi^{-1}(j)$. A *descent* in a permutation π is an index $i \in [p-1]$ such that $\pi(i) > \pi(i+1)$. Let $\mathrm{des}(\pi)$ denote the number of descents in π. The W-polynomial of P is defined by

$$W(P,t) = \sum_{\pi \in \mathcal{L}(P)} t^{\mathrm{des}(\pi)}.$$

The W-polynomials appear naturally in many combinatorial structures, see [2, 7, 8], and are connected to Hilbert series of the Stanley-Reisner rings of simplicial complexes [10, Section III.7] and algebras with straightening laws [9, Thm. 5.2.].

Example 1. Let P_2 be as in Fig. 1. Then

$$\mathcal{L}(P_2) = \{(1,3,2,4),(1,3,4,2),(3,1,2,4),(3,1,4,2),(3,4,1,2)\}$$

so $W(P_2,t) = 4t + t^2$.

When P has no relations, that is, when P is the anti-chain on $[p]$, then $W(P,t)$ is the pth *Eulerian polynomial.* The Eulerian polynomials are known [3] to have all zeros real. This is an instance where the Neggers-Stanley conjecture, also known as the Poset conjecture, holds:

Conjecture 1 (Neggers-Stanley)**.** *Let P be a poset on $[p]$. Then all zeros of $W(P,t)$ are real.*

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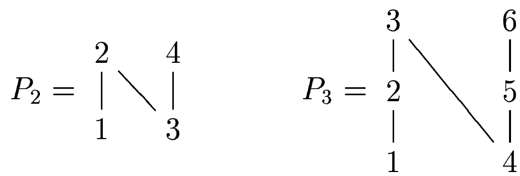
FIGURE 1. The posets P_2 and P_3.



A poset P is naturally labeled if $i \prec j$ implies $i < j$. The above conjecture was made by J. Neggers [4] in 1978 for naturally labeled posets and by R.P. Stanley in 1986 for arbitrary posets on $[p]$. It has been proved for some special cases, see [1, 2, 6, 11].

If a polynomial $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ with non-negative coefficients is real-rooted it follows that the sequence $\{a_i\}_{i=0}^n$ is *unimodal* i.e., there is a d such that $a_0 \leq a_1 \leq \cdots \leq a_{d-1} \leq a_d \geq a_{d+1} \geq \cdots \geq a_n$. This consequence of the Neggers-Stanley conjecture was recently proved [6] (see also [1]) for an important class of naturally labeled posets, namely *graded posets*. A naturally labeled poset P is graded if all maximal chains in P have the same length.

2. Counterexamples to the Neggers-Stanley conjecture

Let $\mathbf{n} \sqcup \mathbf{n}$ be the disjoint union of the chains $1 \prec 2 \prec \cdots \prec n$ and $n{+}1 \prec n{+}2 \prec \cdots \prec 2n$ and let P_n be the poset obtained by adding the relation $n+1 \prec n$ to the relations in $\mathbf{n} \sqcup \mathbf{n}$, see Fig. 1. The only linear extension of $\mathbf{n} \sqcup \mathbf{n}$ which is not a linear extension of P_n is $(1, 2, \ldots, 2n)$, which gives

$$W(\mathbf{n} \sqcup \mathbf{n}, t) = 1 + W(P_n, t).$$

Let π be a permutation. We say that $\pi(i)$ is a *descent top* and that $\pi(i+1)$ is a *descent bottom* if i is a descent in π. A permutation in $\mathcal{L}(\mathbf{n} \sqcup \mathbf{n})$ is uniquely determined by its descent tops (which are necessarily elements of $[2n] \setminus [n]$) and its descent bottoms (which are elements of $[n]$). It follows that the number of permutations in $\mathcal{L}(\mathbf{n} \sqcup \mathbf{n})$ with exactly k descents is $\binom{n}{k}\binom{n}{k}$, so we have

$$W(\mathbf{n} \sqcup \mathbf{n}, t) = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{k} t^k,$$

and hence

$$W(P_n, t) = \sum_{k=1}^{n} \binom{n}{k}\binom{n}{k} t^k.$$

When $n = 11$, Sturm's Theorem [5, Section 10.5] can be used to show that $W(P_n, t)$ has two non-real zeros. These are approximately

$$z = -0.10902 \pm 0.01308i.$$

It is known [7] that $W(\mathbf{n} \sqcup \mathbf{n}, t)$ has real and simple zeros only. One explanation for why $W(P_{11}, t)$ fails to have only real zeros is that adding a constant of sufficiently large modulus to a real- and simple-rooted polynomial of degree greater than 2 destroys the property of being real-rooted.

A counter-example with a polynomial of lower degree can be obtained as follows. Let $P_{n,m}$ be the disjoint union of the chains $1 \prec 2 \prec \cdots \prec n$ and $n+1 \prec n+2 \prec \cdots \prec n+m$ with the relation $n+1 \prec n$ added. Then the W-polynomial of $P_{n,m}$ is given by

$$W(P_{n,m}, t) = \sum_{k \geq 1} \binom{n}{k} \binom{m}{k} t^k.$$

We have

$$W(P_{36,6}, t) = 216t + 9450t^2 + 142800t^3 + 883575t^4 + 2261952t^5 + 1947792t^6.$$

This polynomial has two non-real zeros.

Remark 1. It should be noted that our counterexamples are not naturally labeled, since we have $n+1 \prec n$. The Neggers-Stanley conjecture is therefore still open for naturally labeled posets.

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